

Numerical Methods - I

solⁿ of algebraic and transcendental Eq^s: Regula-Falsh method and Newton-Raphson method formulae and problems.

Finite differences, Interpolation using Newton's forward and backward difference formulae, Newton's divided difference formulae and Lagrange's interpolation formula and problems. Numerical integraⁿ: Trapezoidal, Simpson's (1/3)rd & (3/8)th rules.

Algebraic Eq^s $\Rightarrow$

Eq^s involving algebraic quantities like x, x^2, x^3 are called as algebraic Eq^s.

Ex: i) $x^4 + x^3 = 80$ ii) $x^3 - 4x - 9 = 0$

Transcendental Eq^s $\Rightarrow$

Eq^s involving non-algebraic quantities like $e^x, \log x, \sin x, \tan x$ etc are called as transcendental Equations.

Ex: i) $xe^x - 2 = 0$ ii) $x \cdot \log x - 12 = 0$

Iteration process $\Rightarrow$ Numerical methods of finding approximate roots of the given Eqⁿ is a repetitive type of process known as iterative process.

Fundamental property $\Rightarrow$

If $f(x) = 0$ is a real valued continuous funⁿ of the real variable x . we have the following fundamental property:

If there exist two values a, b

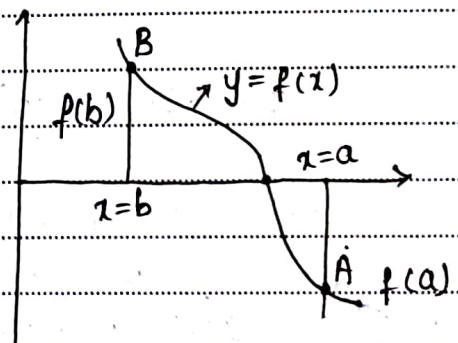
such that $f(x)$ have (different) opposite signs.

Say $f(a) < 0$, $f(b) > 0$ Equivalently $f(a) \cdot f(b) < 0$ then there always exists atleast one real root in the interval (a, b)

There are two numerical iterative methods

i) Newton-Raphson method.

ii) Regula-Falsh method.



Newton-Raphson method $\Rightarrow$

The first approximation to the root x_0 is given by

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

The second approximation is obtained by replacing x_0 by x_1 in the RHS of the above expression.

In general, $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ is called the Newton-Raphson iteration formula.

This procedure is repeated upto two consecutive approximated roots become equal.

Problem $\Rightarrow$

- 1) Find the real root of the equation $x^3 + x^2 + 3x + 4 = 0$ by applying Newton-Raphson method (Carryout 2 iterations).

Solⁿ: Let $f(x) = x^3 + x^2 + 3x + 4$

$$f(0) = 4, f(-1) = 1, f(-2) = -6 < 0$$

$\therefore$ A real root lies in $(-2, -1)$

$$\text{Let } x_0 = -1. \text{ Also } f'(x) = 3x^2 + 2x + 3$$

$$\text{I}^{\text{st}} \text{ Approximation: } x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = -1 - \frac{f(-1)}{f'(-1)} = -1.25$$

$$\text{II}^{\text{nd}} \text{ Approximation: } x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = -1.25 - \frac{f(-1.25)}{f'(-1.25)}$$

$$x_2 = -1.2229$$

$\therefore$ Thus the required real root is $-1.2229 \approx -1.223$

- 2) Use Newton-Raphson method to find a real root of the Eqⁿ $x^3 - 2x - 5 = 0$. correct to three decimal places.

01st: let $f(x) = x^3 - 2x - 5$

$$f(0) = -5 < 0, f(1) = -6 < 0, f(2) = -1 < 0, f(3) = 16 > 0$$

A root lies in $(2, 3)$

$$\text{Let } x_0 = 2, f'(x) = 3x^2 - 2$$

$$\text{I}^{\text{st}} \text{ approximation: } x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 2 - \frac{f(2)}{f'(2)} = 2.1$$

#1 IInd Approximation: $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 2.1 - \frac{f(2.1)}{f'(2.1)} = 2.0946$

IIIrd Approximation: $x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 2.0946 - \frac{f(2.0946)}{f'(2.0946)} = 2.0946$

Thus the required approximate root correct to three decimal places is 2.095.

3) Use Newton-Raphson method to find a real root of $x \sin x + \cos x = 0$ near $x = \pi$. Carry out the iterations upto four decimal places of accuracy.

solⁿ: Let $f(x) = x \sin x + \cos x$

Let $x_0 = \pi$, $f'(x) = x \cos x + \sin x - \sin x = x \cos x$

$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = \pi - \frac{f(\pi)}{f'(\pi)} = \pi - \frac{(\pi \sin \pi + \cos \pi)}{\pi \cos \pi} = \pi - \frac{1}{\pi} = 2.8233$

$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 2.7986 \checkmark$

$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 2.7984$

$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} = 2.7984$

Thus the required real root is 2.7984

$$\begin{array}{r} 0.0662 \\ -2.6814 \\ \hline = 0.02968 \end{array}$$

4) Find the real root of the Equation $x \cdot e^x - 2 = 0$ correct to three decimal using Newton-Raphson method.

solⁿ: let $f(x) = x e^x - 2$

$f(0) = -2 < 0$, $f(1) = 0.7183 > 0$

$\therefore$ a real root lies in $(0, 1)$

Let $x_0 = 1$, $f'(x) = x e^x + e^x = e^x(x+1)$

$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1 - \frac{f(1)}{f'(1)} = 1 - \frac{0.7183}{e^2} = 0.8679$

$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 0.8528$

$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 0.8526$

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} = 0.8526$$

Thus the required real root correct to three decimal places is 0.853.

54. Using Newton-Raphson method find an approximate root of the Eq.ⁿ $x \log_{10} x = 1.2$ correct to 5 decimal places that near 2.5

Sol.ⁿ: Let $f(x) = x \log_{10} x - 1.2$ $x_0 = 2.5$

$$f(x) = \frac{x \log_e x}{\log_{10} e} - 1.2$$

$$f'(x) = \frac{1}{\log_{10} e} [1 + \log_e x]$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 2.5 - \frac{f(2.5)}{f'(2.5)} = 2.7465$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 2.74065$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 2.74065$$

$\therefore$ The required approximate root is 2.74065

Regula-Falsi method or method of False position $\Rightarrow$

To find the numerical solⁿ of the Eq.ⁿ $f(x) = 0$ by Regula-Falsi method or method of false position.

Let the root of the equation $f(x) = 0$ lies between a and b then $f(a)$ and $f(b)$ are of opposite sign

$\therefore$ The first approximation of the root is given by

$$x = \frac{a f(b) - b f(a)}{f(b) - f(a)}$$

The procedure is repeated upto two consecutive approximate root becomes equal.

It is important to note that if the values a, b of x are nearer enough then the numerical iterative methods will give the real root to the desired accuracy quickly

Problems $\Rightarrow$

- 1) Use the Regula-falsi method to find a real root of the Equation $x^3 - 2x - 5 = 0$ correct to three decimal places.

Solⁿ:

Let $f(x) = x^3 - 2x - 5$

$f(0) = -5$, $f(1) = -6$, $f(2) = -1$, $f(3) = 16$

$\therefore$ The root lies in $(2, 3)$

Since -1 is nearer to zero we can take the initial approximation. x_0 as nearer to 2.

Let $f(2.1) = 0.061 > 0$, $\therefore f(2)$ is $-ve$ & $f(2.1)$ is $+ve$
Root lies betⁿ $(2, 2.1)$

Ist approximation: $x_1 = \frac{a f(b) - b f(a)}{f(b) - f(a)}$

$a = 2$: $f(a) = f(2) = -1$

$b = 2.1$: $f(b) = f(2.1) = 0.061$

$\therefore x_1 = 2.0942$

$f(x_1) = f(2.0942) = -0.00392 < 0$

$\therefore$ Root lies betⁿ $(2.0942, 2.1)$

IInd approximation $\therefore x_2 = \frac{a f(b) - b f(a)}{f(b) - f(a)}$

$a = 2.0942$: $f(a) = f(2.0942) = -0.00392$

$b = 2.1$: $f(b) = f(2.1) = 0.061$

$x_2 = 2.09455$

Since upto 3 decimal places $x_1 = x_2 = 2.094$

Thus the required approximate root is 2.094

(2)

Find the real root of the Equation $\tan x + \tanh x = 0$ lies between 2 and 3 by using Regula-falsi method Carry out 3-iterations.

Solⁿ:

Let $f(x) = \tan x + \tanh x$

$f(2) = \tan 2 + \tanh 2 = -1.221 < 0$

$f(3) = \tan 3 + \tanh 3 = 0.8525 > 0$

Ist Approximation: $x_1 = \frac{a f(b) - b f(a)}{f(b) - f(a)}$

$$a = 2, f(a) = f(2) = -1.221$$

$$b = 3, f(b) = f(3) = 0.8525$$

$$x_1 = 2.59$$

$$f(x_1) = f(2.59) = 0.3735 > 0$$

$\therefore$ Roots lies in $(a, 2.59)$

IInd Approximation : $x_2 = \frac{a f(b) - b f(a)}{f(b) - f(a)}$

$$a = 2, f(a) = f(2) = -1.221$$

$$b = 2.59, f(b) = f(2.59) = 0.3735$$

$$x_2 = 2.4518$$

$$f(x_2) = f(2.4518) = 0.1603 > 0$$

$\therefore$ Roots lies in $(a, 2.4518)$

IIIrd Approximation : $x_3 = \frac{a f(b) - b f(a)}{f(b) - f(a)}$

$$a = 2, f(a) = f(2) = -1.221$$

$$b = 2.4518, f(b) = f(2.4518) = 0.1603$$

$$x_3 = 2.3994$$

$\therefore$ Thus the required third approximation is 2.3994

37 Use the Regula-Falsi method to obtain a root of the Eqⁿ
 $2x - \log_{10} x = 7$ which lies betⁿ 3.5 and 4.

Solⁿ: Let $f(x) = 2x - \log_{10} x - 7$

$$f(3.5) = -0.5441 < 0, f(4) = 0.3979 > 0$$

$\therefore$ Roots lies betⁿ (3.5, 4)

$$a = 3.5, f(a) = f(3.5) = -0.5441$$

$$b = 4, f(b) = f(4) = 0.3979$$

Ist Approximation : $x_1 = \frac{a f(b) - b f(a)}{f(b) - f(a)} = 3.79$

$$f(x_1) = f(3.79) = 0.00136 > 0$$

$\therefore$ Root lies in (3.5, 3.79)

IInd Approximation : $x_2 = \frac{a f(b) - b f(a)}{f(b) - f(a)}$

$$a = 3.5, f(a) = f(3.5) = -0.5441$$

$$b = 3.79, f(b) = f(3.79) = 0.00136$$

$$x_2 = 3.7893 \approx 3.79$$

Thus the approximate root correct to two decimal places is 3.79

47 Find the smallest positive root of the equation $x^2 - \log_e x = 12$ by Regula-Falsi method.

Soln: Let $f(x) = x^2 - \log_e x - 12$

$$f(1) = -11 < 0, f(2) = -8.69 < 0, f(3) = -4.0986 < 0, f(4) = 2.6137 > 0$$

$\therefore$ A real root lies in (3, 4) and will be in the neighbourhood of 4.

$$\therefore f(3.6) = -0.3209, f(3.7) = 0.3817$$

$\therefore$ Root lies in (3.6, 3.7)

Ist Approximation $\Rightarrow x_1 = \frac{a f(b) - b f(a)}{f(b) - f(a)}$

$$a = 3.6, f(3.6) = -0.3209; b = 3.7, f(3.7) = 0.3817$$

$$x_1 = 3.6457$$

$$f(x_1) = -0.00242 < 0$$

$\therefore$ Root lies in (3.6457, 3.7)

IInd Approximation $\Rightarrow x_2 = \frac{a f(b) - b f(a)}{f(b) - f(a)}$

$$a = 3.6457, f(a) = -0.00242; b = 3.7, f(b) = 0.3817$$

$$x_2 = 3.646042$$

Thus the required root correct to 3 decimal places is 3.646

3.5425

57 Find the root of the equation $x \cdot e^x - \cos x = 0$ by the method of false position.

Soln: Let $f(x) = x \cdot e^x - \cos x$

$$f(0) = -1 < 0, f(1) = 2.178 > 0$$

$$f(0.5) = -0.0532 < 0, f(0.6) = 0.2679 > 0$$

$\therefore$ Root lies in (0.5, 0.6)

Ist Approximation $\Rightarrow x_1 = \frac{a f(b) - b f(a)}{f(b) - f(a)}$

$$a = 0.5, f(0.5) = -0.0532$$

$$b = 0.6, f(0.6) = 0.2679$$

$$x_1 = 0.5166$$

$$f(x_1) = f(0.5166) = -0.0035 < 0$$

∴ Root lies in $(0.5166, 0.6)$

IInd Approximation : $x_2 = \frac{af(b) - bf(a)}{f(b) - f(a)}$

$$a = 0.5166 \quad f(0.5166) = -0.0035$$

$$b = 0.6 \quad f(0.6) = 0.2679$$

$$x_2 = 0.5177$$

$$f(x_2) = f(0.5177) = -0.00017 < 0$$

∴ Root lies in $(0.5177, 0.6)$

IIIrd Approximation : $x_3 = \frac{af(b) - bf(a)}{f(b) - f(a)}$

$$a = 0.5177 \quad f(a) = -0.00017$$

$$b = 0.6 \quad f(b) = 0.2679$$

$$x_3 = 0.5178$$

Thus the required root is 0.5178

Finite Difference ⇒

Let $y = f(x)$ be a funⁿ, in this function 'x' is independent variable and 'y' is dependent variable. By assigning a value for x, we get a corresponding value $y = f(x)$ for the funⁿ. The value assigned to x is called the argument. The corresponding value of the funⁿ is called entry. The difference betⁿ 2 consecutive values of x is called the interval of difference. It is denoted by h.

Forward Difference ⇒

Let $y = f(x)$ be a funⁿ & 'h' is the interval of difference, then $f(x+h) - f(x)$ is called the first forward difference of the funⁿ between the arguments x and x+h. It is denoted by $\Delta f(x)$

$$\Delta f(x) = f(x+h) - f(x)$$

$$\Delta f(x_0) = f(x_0+h) - f(x_0) ; \Delta y_0 = y_1 - y_0$$

$$\Delta f(x_1) = f(x_1+h) - f(x_1) ; \Delta y_1 = y_2 - y_1$$

⋮

$$\Delta f(x_n) = f(x_n + h) - f(x_n) = \Delta y_{n-1} = y_n - y_{n-1}$$

The difference of the first forward differences are called second forward differences

$$\text{i.e. } \Delta^2 y_0 = \Delta y_1 - \Delta y_0$$

$$\Delta^2 y_1 = \Delta y_2 - \Delta y_1$$

$$\Delta^2 y_{n-2} = \Delta y_{n-1} - \Delta y_{n-2}$$

Forward Difference table $\Rightarrow$

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	...	$\Delta^n y$
x_0	y_0	Δy_0				
x_1	y_1	Δy_1	$\Delta^2 y_0$	$\Delta^3 y_0$		
x_2	y_2	Δy_2	$\Delta^2 y_1$			
x_3	y_3				...	$\Delta^n y_0$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\Delta^3 y_{n-3}$		
x_{n-1}	y_{n-1}	Δy_{n-1}	$\Delta^2 y_{n-2}$			
x_n	y_n					

The first entries in the table $\Delta y_0, \Delta^2 y_0, \Delta^3 y_0, \dots, \Delta^n y_0$ are called the leading forward Differences

Backward Difference $\Rightarrow$

Let $f(x)$ be a given funⁿ and h be the difference of interval, then the backward difference ∇ (nabla) is defined as $\nabla f(x) = f(x) - f(x-h)$

$$\nabla f(x_n) = f(x_n) - f(x_n - h) : \nabla y_n = y_n - y_{n-1}$$

$$\nabla f(x_{n-1}) = f(x_{n-1}) - f(x_{n-1} - h) : \nabla y_{n-1} = y_{n-1} - y_{n-2}$$

$$\nabla y_2 = y_2 - y_1$$

$$\nabla y_1 = y_1 - y_0$$

Similarly second backward difference are defined as

$$\nabla^2 y_n = \nabla y_n - \nabla y_{n-1}$$

$$\nabla^2 y_{n-1} = \nabla y_{n-1} - \nabla y_{n-2}$$

$$\nabla^2 y_2 = \nabla y_2 - \nabla y_1$$

Backward Difference table $\Rightarrow$

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$		$\nabla^n y$
x_0	y_0	∇y_1	$\nabla^2 y_2$	$\nabla^3 y_2$		
x_1	y_1	∇y_2	$\nabla^2 y_3$	$\nabla^3 y_3$		
x_2	y_2	∇y_3	$\nabla^2 y_4$	$\nabla^3 y_4$		
x_3	y_3	∇y_4	$\nabla^2 y_5$	$\nabla^3 y_5$		
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$		
x_{n-1}	y_{n-1}	∇y_n	$\nabla^2 y_n$	$\nabla^3 y_n$		
x_n	y_n					

The last entries in the table $\nabla y_n, \nabla^2 y_n, \nabla^3 y_n, \dots, \nabla^n y_n$ are called the leading backward differences.

Interpolation $\Rightarrow$

The method of finding the value of 'y' at a specified value of 'x', having known values of x and y is called Interpolation.

Interpolation $\Rightarrow$

If $y_0, y_1, y_2, \dots, y_n$ be a set of values of an unknown funⁿ $y=f(x)$ corresponding to the values of $x: x_0, x_1, x_2, \dots, x_n$, the process of finding the value of y for any given value of x betⁿ x_0 and x_n is called interpolation. Also the process of finding the value of y outside the given range of x is called extrapolation. In general the concept of interpolation includes extrapolation also.

Hence we can say that interpolation is a technique of estimating the value of an unknown function for any intermediate value of the independent variable.

For ex: If we have the data

x	0	4	5	8	10	15
$f(x)$	6	15	17	29	40	87

The process of estimating $f(3)$, $f(7)$, $f(12.5)$, $f(14)$ etc. is interpolation. and estimating $f(-0.5)$, $f(18)$, $f(25)$ etc. is Extrapolation.

Interpolation formulae for equal intervals / equidistant arguments : $\Rightarrow$

We discuss interpolation formulae for equal intervals based on forward and backward differences. These formulae are established by approximating the unknown fun.ⁿ to a polynomial in x whose values coincide with the values of $f(x)$ at the specified points of $x: x_0, x_1, x_2, \dots, x_n$.

Newton's forward interpolation formula :

The value of $y = f(x)$ at $x = x_0 + rh$ that is $y_r = f(x_0 + rh)$ is approximately given by

$$y_r = y_0 + r \Delta y_0 + \frac{r(r-1)}{2!} \Delta^2 y_0 + \frac{r(r-1)(r-2)}{3!} \Delta^3 y_0 + \dots + \frac{r(r-1)(r-2) \dots (r-n+1)}{n!} \Delta^n y_0$$

where r is any real number.

Newton's backward interpolation formula :

The value of $y = f(x)$ at $x = x_n + rh$ that is $y_r = f(x_n + rh)$ is approximately given by

$$y_r = y_n + r \nabla y_n + \frac{r(r+1)}{2!} \nabla^2 y_n + \frac{r(r+1)(r+2)}{3!} \nabla^3 y_n + \dots + \frac{r(r+1)(r+2) \dots (r+n-1)}{n!} \nabla^n y_n$$

The polynomial $y = f(x)$ satisfying the data can also be found from these formulae and it is called an interpolating polynomial.

1) Find $y(1.4)$ given that

x	1	2	3	4	5
y	10	26	58	112	194

Sol:-

Here we have to find y at $x = 1.4$. Since the value of $x = 1.4$ is in the first half of the table near $x = 1$.

Newton's forward interpolation formula is appropriate and we shall construct the forward difference table.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1	10	16			
2	26	32	16		
3	58	54	22	6	
4	112	82	28	6	0
5	194				

We have Newton's forward interpolation formula

$$y_r = y_0 + r \cdot \Delta y_0 + \frac{r(r-1)}{2!} \Delta^2 y_0 + \frac{r(r-1)(r-2)}{3!} \Delta^3 y_0 + \dots$$

where $r = \frac{x - x_0}{h}$

x = point at which y is required = 1.4

x_0 = initial point (first value of x) = 1

h = common interval length = 1 Here $r = \frac{1.4 - 1}{1} = 0.4$

From the table, $\Delta y_0 = 16$, $\Delta^2 y_0 = 16$, $\Delta^3 y_0 = 6$, $\Delta^4 y_0 = 0$

$$\therefore y(1.4) = f(1.4) = 10 + (0.4)16 + \frac{(0.4)(0.4-1)}{2} (16) +$$

$$\frac{(0.4)(0.4-1)(0.4-2)}{6} (6)$$

$$= 10 + 6.4 + (0.4)(-0.6)8 + (0.4)(-0.6)(-1.6) = 14.864$$

Thus $y(1.4) = 14.864$

2) Find $u_{0.5}$ from the data.

$$u_0 = 225, u_1 = 238, u_2 = 320, u_3 = 340$$

1) The value $x = 0.5$ is near to $x = 0$ and hence Newton's forward interpolation formula is appropriate.

we shall first construct the forward difference table:

x	$u_x = y$	Δy	$\Delta^2 y$	$\Delta^3 y$
$x_0 = 0$	$y_0 = 225$	$\Delta y_0 = 13$	$\Delta^2 y_0 = 69$	$\Delta^3 y_0 = -131$
1	238	82	$\Delta^2 y_1 = 62$	
2	320	20		
3	340			

we have newton's forward interpolation formula.

$$y_r = y_0 + r \Delta y_0 + \frac{r(r-1)}{2!} \Delta^2 y_0 + \frac{r(r-1)(r-2)}{3!} \Delta^3 y_0 + \dots$$

where $r = \frac{x - x_0}{h}$, $r = \frac{0.5 - 0}{1} = 0.5$

$$\therefore f(0.5) = 225 + 0.5(13) + \frac{(0.5)(0.5-1)}{2} (69) + \frac{(0.5)(0.5-1)(0.5-2)}{6} (-131)$$

$$f(0.5) = 214.6875$$

$$u_{0.5} = 214.6875$$

3) The area of a circle (A) corresponding to diameter (D) is given below

D	80	85	90	95	100
A	5026	5674	6362	7088	7854

Find the area corresponding to diameter 105 using an appropriate interpolation formula.

Sol: Here we have to find A when $D = 105$

As this value 105 is near to the end value 100, Newton's backward interpolation formula is appropriate.

D and A correspond to x and y .

The backward difference table is formed first

$x = D$	$y = A$	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
80	5026	648			
85	5674	688	40		
90	6362	726	38	-2	
95	7088	766	40	-2	4
$x_n = 100$	$y_n = 7854$				

we have newton's backward interpolation formula

$$y_r = y_0 + r \nabla y_0 + \frac{r(r+1)}{2!} \nabla^2 y_0 + \frac{r(r+1)(r+2)}{3!} \nabla^3 y_0 + \frac{r(r+1)(r+2)(r+3)}{4!} \nabla^4 y_0 + \dots$$

where $r = \frac{x - x_0}{h}$; $r = \frac{105 - 100}{5} = +1$

From the table, $\nabla y_0 = 766$, $\nabla^2 y_0 = 40$, $\nabla^3 y_0 = 2$, $\nabla^4 y_0 = 4$
 $\therefore f(105) = 7854 + (1)(766) + \frac{(1)(2)}{2} (40) + \frac{(1)(2)(3)}{6} (2) +$

$$\frac{(1)(2)(3)(4)}{24} (4)$$

$$= 7854 + 766 + 40 + 2 + 4 = 8666$$

Then the area (A) corresponding to diameter (D) = 105
 is 8666

4) The following table give the values of $\tan x$ for $0.10 \leq x \leq 0.30$. Find $\tan(0.26)$

x	0.10	0.15	0.20	0.25	0.30
$\tan x$	0.1003	0.1511	0.2027	0.2553	0.3093

sol: Here the value $x = 0.26$ is near the end value 0.30.
 Hence Newton's backward interpolation formula is appropriate.
 The backward difference table is as follows.

x	$f(x) = y$	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
0.10	0.1003				
		0.0508			
0.15	0.1511		0.0008		
		0.0516		0.0002	
0.20	0.2027		0.0010		0.0002
		0.0526		0.0004	
0.25	0.2553		0.0014		
		0.0540			
0.30	$y_0 = 0.3093$				

we have Newton's backward interpolation formula

$$y_r = y_n + r \nabla y_n + \frac{r(r+1)}{2!} \nabla^2 y_n + \frac{r(r+1)(r+2)}{3!} \nabla^3 y_n + \dots + \frac{r(r+1)(r+2)(r+3)}{4!} \nabla^4 y_n + \dots$$

where $r = \frac{x - x_n}{h}$, $r = \frac{0.26 - 0.30}{0.05} = -0.8$

from the table :

$$\nabla y_n = 0.0540, \nabla^2 y_n = 0.0014, \nabla^3 y_n = 0.0004, \nabla^4 y_n = 0.0002$$

$$\begin{aligned} \therefore f(0.26) &= 0.3093 + (-0.8)(0.054) + \frac{(-0.8)(-0.8+1)(0.0014)}{2} \\ &\quad + \frac{(-0.8)(-0.8+1)(-0.8+2)(0.0004)}{6} \\ &\quad + \frac{(-0.8)(-0.8+1)(-0.8+2)(-0.8+3)(0.0002)}{24} \end{aligned}$$

$$f(0.26) = 0.26602$$

Thus $\tan(0.26) = 0.2660$

5) Extrapolate for 25.4 given the data -

x	19	20	21	22	23
y	91	100.25	110	120.25	131

Solⁿ: we shall first construct the backward difference table

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$
19	91			
		9.25		
20	100.25		0.5	
		9.75		0
21	110		0.5	
		10.25		0
22	120.25		$\nabla^2 y_n = 0.5$	$\nabla^3 y_n$
		10.75		
$x_n = 23$	$y_n = 131$	∇y_n		

we have newton's backward interpolation formula.

$$y_r = y_n + r \nabla y_n + \frac{r(r+1)}{2!} \nabla^2 y_n + \frac{r(r+1)(r+2)}{3!} \nabla^3 y_n + \dots$$

where $r = \frac{x - x_n}{h}$, $r = \frac{25.4 - 23}{1} = 2.4$

$$\therefore f(25.4) = 131 + (2.4)10.75 + \frac{(2.4)(2.4+1)(0.5)}{2}$$

$$f(25.4) = 158.84$$

6) Given $f(40) = 184$, $f(50) = 204$, $f(60) = 226$, $f(70) = 250$, $f(80) = 276$, $f(90) = 304$. find $f(38)$ & $f(85)$ using suitable interpolation formulae.

Sol: Here we shall find $f(38)$ using Newton's forward interpolation formula and find $f(85)$ using Newton's backward interpolation formula. The finite difference table applicable to both the interpolation formulae is as follows.

x	$f(x) = y$	I Difference	II Difference	III Difference
40	184	20		
50	204	22	2	
60	226	24	2	0
70	250	26	2	0
80	276	28	2	0
90	304			

To find $f(38)$: we have from the table

$$x_0 = 40, y_0 = 184, \Delta y_0 = 20, \Delta^2 y_0 = 2, \Delta^3 y_0 = 0$$

we have Newton's forward interpolation formula.

$$y_r = y_0 + r \Delta y_0 + \frac{r(r-1)}{2!} \Delta^2 y_0 + \dots$$

$$\text{where } r = \frac{x - x_0}{h} : r = \frac{38 - 40}{10} = -0.2$$

$$f(38) = 184 + (-0.2)(20) + \frac{(-0.2)(-0.2-1)}{2} (2) = 180.24$$

$$\text{thus } f(38) = 180.24$$

To find $f(85)$: we have from the table

$$x_n = 90, y_n = 304, \nabla y_n = 28, \nabla^2 y_n = 2, \nabla^3 y_n = 0$$

we have Newton's backward interpolation formula.

$$y_r = y_n + r \nabla y_n + \frac{r(r+1)}{2!} \nabla^2 y_n + \dots$$

where $r = \frac{x - x_n}{h}$: $r = \frac{85 - 90}{10} = -0.5$

$$f(85) = 304 + (-0.5)(28) + \frac{(-0.5)(-0.5+1)(2)}{2}$$

$$f(85) = 289.75$$

7> Given $\sin 45^\circ = 0.7071$, $\sin 50^\circ = 0.7660$, $\sin 55^\circ = 0.8192$, $\sin 60^\circ = 0.8660$, find $\sin 57^\circ$ using an appropriate interpolation formula.

soln: we have to find the value of $f(x) = \sin x$ at $x = 57^\circ$ which is near the end value $x = 60^\circ$ and hence Newton's backward interpolation formula is appropriate.

The difference table is as follows

x	$f(x) = y$	∇y	$\nabla^2 y$	$\nabla^3 y$
45	0.7071			
50	0.7660	0.0589		
55	0.8192	0.0532	-0.0057	
60	0.8660	0.0468	-0.0064	-0.0007

we have Newton's backward interpolation formula.

$$y_r = y_n + r \nabla y_n + \frac{r(r+1)}{2!} \nabla^2 y_n + \frac{r(r+1)(r+2)}{3!} \nabla^3 y_n + \dots$$

where $r = \frac{x - x_n}{h}$: $r = \frac{57 - 60}{5} = -0.6$

From the table: $\nabla y_n = 0.0468$; $\nabla^2 y_n = -0.0064$

$$\nabla^3 y_n = -0.0007$$

$$\therefore f(57) = 0.8660 + (-0.6)(0.0468) + \frac{(-0.6)(-0.6+1)(-0.0064)}{2}$$

$$+ \frac{(-0.6)(-0.6+1)(-0.6+2)(-0.0007)}{6} = 0.8387$$

Thus $\sin 57^\circ = 0.8387$

8> Find the interpolating polynomial $f(x)$ satisfying $f(0) = 0$, $f(2) = 4$, $f(4) = 56$, $f(6) = 204$, $f(8) = 496$, $f(10) = 980$ and hence find $f(3)$, $f(5)$, $f(7)$

Solⁿ: The interpolating polynomial can be found from either of the two interpolation formulae. We shall use Newton's forward interpolation formula.

x	$f(x)=y$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
$x_0=0$	$y_0=0$	$\Delta y_0=4$	$\Delta^2 y_0=48$	$\Delta^3 y_0=48$	$\Delta^4 y_0=0$
2	4	52	96	48	0
4	56	148	144	48	0
6	204	292	192		
8	496	484			
10	980				

we now have Newton's forward interpolation formula:

$$y_r = y_0 + r \Delta y_0 + \frac{r(r-1)}{2!} \Delta^2 y_0 + \frac{r(r-1)(r-2)}{3!} \Delta^3 y_0 + \dots$$

where $r = \frac{x-x_0}{h}$ Here, $r = \frac{x-0}{2}$ or $r = \frac{x}{2}$

$$\therefore y = f(x) = 0 + \frac{x}{2}(4) + \frac{(x/2)(x/2-1)}{2}(48) + \frac{(x/2)(x/2-1)(x/2-2)}{6}(48)$$

$$= 2x + \frac{x}{2} \left(\frac{x-2}{2} \right) (24) + \frac{x}{2} \left(\frac{x-2}{2} \right) \left(\frac{x-4}{2} \right) (8)$$

$$= 2x + x(x-2)(6) + x(x-2)(x-4)$$

$$= x[2 + 6(x-2) + (x^2 - 2x - 4x + 8)]$$

$$y = f(x) = x[2 + 6x - 12 + x^2 - 6x + 8] = x^3 - 2x$$

Thus the interpolating polynomial is $y = f(x) = x^3 - 2x$

Now by putting $x = 3, 5, 7$ we obtain.

$$f(3) = 21, f(5) = 115, f(7) = 329$$

Q5 From the following table find the number of students who have obtained at least 45 marks

b) betⁿ 40 and 45 marks

Marks	30-40	40-50	50-60	60-70	70-80
No. of Students	31	42	51	35	31

Solⁿ: we shall reconstitute the given table with $f(x)$ representing the number of students less than x marks. That is

less than 40 marks 31 students

less than 50 marks $31 + 42 = 73$ students

less than 60 marks $73 + 51 = 124$ students

less than 70 marks $124 + 35 = 159$ students

less than 80 marks $159 + 31 = 190$ students

we have the new table along with forward differences.

a) we need to find $f(45)$ being the number of students scoring less than 45 marks

x	$f(x) = y$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
$x_0 = 40$	$y_0 = 31$	$\Delta y_0 = 42$	$\Delta^2 y_0 = 9$	$\Delta^3 y_0 = -25$	$\Delta^4 y_0 = 37$
50	73	51	$= -16$	12	
60	124	35	$= -4$		
70	159	31			
80	190				

we shall find $f(45)$ by applying Newton's forward interpolation formula.

$$y_r = y_0 + r \Delta y_0 + \frac{r(r-1)}{2!} \Delta^2 y_0 + \frac{r(r-1)(r-2)}{3!} \Delta^3 y_0 + \dots$$

where, $r = \frac{x - x_0}{h}$: $r = \frac{45 - 40}{10} = 0.5$

$$\therefore f(45) = 31 + (0.5) 42 + \frac{(0.5)(0.5-1)}{2} (9)$$

$$+ \frac{(0.5)(0.5-1)(0.5-2)}{6} (-25) + \frac{(0.5)(0.5-1)(0.5-2)(0.5-3)}{24} (37)$$

$$f(45) = 47.86 \approx 48$$

Thus the number of students obtaining less than 45 marks is 48

b) we need to find $f(45) - f(40)$. But $f(40) = 31$ by data.

$$\text{Hence } f(45) - f(40) = 48 - 31 = 17$$

Thus the number of students scoring marks between 40 and 45 is 17

A Survey conducted in a slum locality reveals the following information as classified below.

Income per day	under 10	10-20	20-30	30-40	40-50
No. of persons	20	45	115	210	115

Estimate the probable no. of persons in the income group of 20 to 25.
 The given data is reconstituted with $f(x)$ representing the number of persons less than income of Rs. x . That is
 less than Rs. 10 = 20

$$\text{less than Rs. 20} = 20 + 45 = 65$$

$$\text{less than Rs. 30} = 65 + 115 = 180$$

$$\text{less than Rs. 40} = 180 + 210 = 390$$

$$\text{less than Rs. 50} = 390 + 115 = 505$$

we have to find $f(25)$ and $f(20)$ which estimates number of persons having income less than Rs. 25 and less than Rs. 20 so that their difference, that is $f(25) - f(20)$ will give us the number of persons having income betⁿ Rs. 20 & 25. The new table along with the forward differences is as follows.

x	$f(x) = y$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
$x_0 = 10$	$y_0 = 20$				
20	65	45			
30	180	115	70		
40	390	210	95	25	
50	505	115	-95	-190	-215

$$r = \frac{x - x_0}{h} \quad ; \quad r = \frac{x - 10}{10}$$

we need to find $f(25) - f(20)$

$$\text{First we shall find } f(25) \Rightarrow r = \frac{25 - 10}{10} = 1.5$$

$$\text{Now } f(25) \approx 107$$

$$\text{from the table } f(20) = 65$$

$$\text{Hence } f(25) - f(20) = 107 - 65 = 42$$

Thus the no. of persons in the income group of Rs. 20 to 25 is 42

- (11) In a table given below, the values of y are consecutive terms of a series of which 23.6 is the 6th term. Find the first and tenth terms of the series.

x	3	4	5	6	7	8	9
y	4.8	8.4	14.5	23.6	36.2	52.8	73.9

solⁿ: we need to compute $y(1)$ & $y(10)$. N.F & B.I.F respectively will be appropriate.

we shall first construct the finite difference table

x	y	I st Diff	II nd Diff	III rd Diff	IV th Diff
3	4.8	3.6			
4	8.4	6.1	2.5		
5	14.5	9.1	3.0	0.5	0
6	23.6	12.6	3.5	0.5	0
7	36.2	16.6	4.0	0.5	0
8	52.8	21.1	4.5		
9	73.9				

Case i) To find $y(1)$

N.F.I.F

$$r = \frac{x - x_0}{h} : r = \frac{1-3}{1} = -2$$

Now, $y(1) = 3.1$

Case ii) To find $y(10)$

$$r = \frac{x - x_n}{h} : r = \frac{10-9}{1} = 1$$

Now, $y(10) = 100$

- (12) The population for a town is given by the following table.

year	1951	1961	1971	1981	1991
pop ⁿ in thou	19.96	39.65	58.81	77.21	94.61

Using Newton's F & B I.F. Calculate the increase in the population from the year 1955 to 1985

solⁿ: we shall first form the finite Diff table

x	y	1 st Diff	2 nd Diff	3 rd Diff	4 th Diff
1951	19.96				
1961	39.65	19.69			
1971	58.81	19.16	-0.53	-0.23	-0.01
1981	77.21	18.4	-0.76	-0.24	
1991	94.61	17.4	-1		

Case i) : To find $y(1955)$

N.F.I.F

$$r = \frac{x - x_0}{h} = \frac{1955 - 1951}{10} = 0.4$$

$$y(1955) = 27.89$$

Case ii) To find $y(1985)$

N.B.I.F

$$r = \frac{x - x_n}{h} = \frac{1985 - 1991}{10} = -0.6$$

$$y(1985) = 84.3$$

Interpolation formulae for unequal intervals

* Divided differences $\Rightarrow$

Let $f(x_0), f(x_1), f(x_2), \dots, f(x_n)$ be the values of an unknown funⁿ $y = f(x)$ corresponding to the values of x $x_0, x_1, x_2, \dots, x_n$ at unequal intervals. The first order Divided differences are defined as follows.

$$f(x_0, x_1) = \frac{f(x_1) - f(x_0)}{x_1 - x_0}, \quad f(x_1, x_2) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

$$f(x_2, x_3) = \frac{f(x_3) - f(x_2)}{x_3 - x_2}, \dots, f(x_{n-1}, x_n) = \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}$$

The second order divided differences are defined as follows

$$f(x_0, x_1, x_2) = \frac{f(x_1, x_2) - f(x_0, x_1)}{x_2 - x_0}$$

$$f(x_1, x_2, x_3) = \frac{f(x_2, x_3) - f(x_1, x_2)}{x_3 - x_1} \text{ etc.}$$

$$f(x_{n-2}, x_{n-1}, x_n) = \frac{f(x_{n-1}, x_n) - f(x_{n-2}, x_{n-1})}{x_n - x_{n-2}}$$

Similarly the other higher order divided differences are defined. The tabular arrangement of these values is called the divided difference table and is as follows.

x	$f(x)$	1st DD	2nd D.D	---	n^{th} DD
x_0	$f(x_0)$				
x_1	$f(x_1)$	$f(x_0, x_1)$	$f(x_0, x_1, x_2)$		
x_2	$f(x_2)$	$f(x_1, x_2)$	$f(x_1, x_2, x_3)$		
---	---	---	---		$f(x_0, x_1, x_2, \dots, x_n)$
x_{n-2}	$f(x_{n-2})$	$f(x_{n-2}, x_{n-1})$	$f(x_{n-3}, x_{n-2}, x_{n-1})$		
x_{n-1}	$f(x_{n-1})$	$f(x_{n-1}, x_n)$	$f(x_{n-2}, x_{n-1}, x_n)$		
x_n	$f(x_n)$				

Note : The notation for divided differences should not be confused with the notation for funⁿs of two or more variables. $f(x_1, x_2)$ does not stand for a funⁿ of two variables x_1 & x_2 ; similarly $f(x_1, x_2, x_3)$ is not a funⁿ of x_1, x_2, x_3 and so on. Here $x_1, x_2, \dots$ are called arguments.

Newton's divided difference formula or Newton's general interpolation formula.

Statement : $\Rightarrow$ If $f(x_0), f(x_1), f(x_2), \dots, f(x_n)$ be a set of values of an unknown funⁿ $f(x)$ corresponding to the values of $x : x_0, x_1, x_2, \dots, x_n$ at unequal intervals then

$$y = f(x) = f(x_0) + (x - x_0) f(x_0, x_1) + (x - x_0)(x - x_1) f(x_0, x_1, x_2) + \dots + (x - x_0)(x - x_1) \dots (x - x_{n-1}) f(x_0, x_1, x_2, \dots, x_n)$$

01) Use Newton's divided difference formula to find $f(4)$ given data :

x	0	2	3	6
$f(x)$	-4	2	14	158

solⁿ:-

we shall first form the table of divided differences

x	$y = f(x)$	1 st DD	2 nd DD	3 rd DD
$x_0 = 0$	$f(x_0) = -4$	$f(x_0, x_1)$ $\frac{2 - (-4)}{2 - 0} = 3$	$f(x_0, x_1, x_2)$ $\frac{12 - 3}{3 - 0} = 3$	$f(x_0, x_1, x_2, x_3)$ $\frac{9 - 3}{6 - 0} = 1$
$x_1 = 2$	$f(x_1) = 2$	$f(x_1, x_2)$ $\frac{14 - 2}{3 - 2} = 12$	$f(x_2, x_3, x_4)$ $\frac{48 - 12}{6 - 2} = 9$	
$x_2 = 3$	$f(x_2) = 14$	$f(x_2, x_3)$ $\frac{158 - 14}{6 - 3} = 48$		
$x_3 = 6$	$f(x_3) = 158$			

we have Newton's divided difference formula.

$$f(x) = f(x_0) + (x - x_0)f(x_0, x_1) + (x - x_0)(x - x_1)f(x_0, x_1, x_2) + (x - x_0)(x - x_1)(x - x_2)f(x_0, x_1, x_2, x_3) + \dots$$

$$\therefore f(4) = -4 + (4 - 0)3 + (4 - 0)(4 - 2)(3) + (4 - 0)(4 - 2)(4 - 3)1$$

$$= -4 + 12 + 24 + 8 = 40$$

Thus $f(4) = 40$

2) Construct the interpolation polynomial for the data given below using Newton's general interpolation formula for divided differences.

x	2	4	5	6	8	10
y	10	96	196	350	868	1746

solⁿ: The divided difference table is as follows.

x	$f(x)$	1 st DD	2 nd DD	3 rd DD
$x_0 = 2$	$f(x_0) = 10$	$f(x_0, x_1)$ $\frac{96 - 10}{4 - 2} = 43$	$f(x_0, x_1, x_2)$ $\frac{100 - 43}{5 - 2} = 19$	$f(x_0, x_1, x_2, x_3)$ $\frac{27 - 19}{6 - 2} = 2$
$x_1 = 4$	$f(x_1) = 96$	$f(x_1, x_2)$ $\frac{196 - 96}{5 - 4} = 100$	$f(x_1, x_2, x_3)$ $\frac{154 - 100}{6 - 4} = 27$	$f(x_1, x_2, x_3, x_4)$ $\frac{35 - 27}{8 - 4} = 2$
$x_2 = 5$	$f(x_2) = 196$	$f(x_2, x_3)$ $\frac{350 - 196}{6 - 5} = 154$	$f(x_2, x_3, x_4)$ $\frac{259 - 154}{8 - 5} = 35$	

$$x_3 = 6 \quad f(x_3) = 350$$

$$\frac{f(x_3, x_4) - f(x_3, x_2)}{8-6} = 259$$

$$\frac{f(x_2, x_3, x_4, x_5) - f(x_2, x_3, x_4, x_2)}{10-5} = 2$$

$$x_4 = 8 \quad f(x_4) = 868$$

$$\frac{f(x_4, x_5) - f(x_4, x_3)}{10-8} = 439$$

$$\frac{f(x_3, x_4, x_5) - f(x_3, x_4, x_2)}{10-6} = 45$$

$$x_5 = 10 \quad f(x_5) = 1746$$

The fourth order differences are zero as third order differences are same. we have Newton's general interpolation formula

$$f(x) = f(x_0) + (x-x_0)f(x_0, x_1) + (x-x_0)(x-x_1)f(x_0, x_1, x_2) + (x-x_0)(x-x_1)(x-x_2)f(x_0, x_1, x_2, x_3) + \dots$$

$$\text{Now, } f(x) = 10 + (x-2)(43) + (x-2)(x-4)(19) + (x-2)(x-4)(x-5)(2)$$

$$= 10 + (x-2)[43 + (19x-76) + (x^2-9x+20)2]$$

$$= 10 + (x-2)(2x^2+x+7) = 2x^3-3x^2+5x-4$$

Then the required interpolating polynomial is given by

$$f(x) = 2x^3 - 3x^2 + 5x - 4$$

3) Fit an interpolating polynomial for the data

$$u_{10} = 355, u_0 = -5, u_8 = -21, u_1 = -14, u_4 = -125$$

by using Newton's general interpolation formula and hence find u_2

Sol: We shall arrange the data taking the values of x in the ascending order along with the corresponding values of u_x just for convenience.

x	$u_x = f(x)$	1st DD	2nd DD	3rd DD
$x_0 = 0$	$f(x_0) = -5$	$\frac{f(x_0, x_1) - f(x_0, x_2)}{1-0} = -9$	$\frac{f(x_0, x_1, x_2) - f(x_0, x_1, x_3)}{4-0} = -7$	$\frac{f(x_0, x_1, x_2, x_3) - f(x_0, x_1, x_2, x_4)}{8-0} = 2$
$x_1 = 1$	$f(x_1) = -14$	$\frac{f(x_1, x_2) - f(x_1, x_3)}{4-1} = -37$	$\frac{f(x_1, x_2, x_3) - f(x_1, x_2, x_4)}{8-1} = 9$	$\frac{f(x_1, x_2, x_3, x_4) - f(x_1, x_2, x_3, x_5)}{10-1} = 2$
$x_2 = 4$	$f(x_2) = -125$	$\frac{f(x_2, x_3) - f(x_2, x_4)}{8-4} = 26$	$\frac{f(x_2, x_3, x_4) - f(x_2, x_3, x_5)}{10-4} = 27$	
$x_3 = 8$	$f(x_3) = -21$	$\frac{f(x_3, x_4) - f(x_3, x_5)}{10-8} = 188$		
$x_4 = 10$	$f(x_4) = 355$			

The fourth order differences are zero since the third order differences are same. We have Newton's general interpolation formula.

$$f(x) = f(x_0) + (x-x_0)f(x_0, x_1) + (x-x_0)(x-x_1)f(x_0, x_1, x_2) + (x-x_0)(x-x_1)(x-x_2)f(x_0, x_1, x_2, x_3) + \dots$$

Now $f(x) = -5 + (x)(-9) + x(x-1)(-7) + (x)(x-1)(x-4)(2)$
 $= -5 + x[-9 + (-7x+7) + (x^2-5x+4)2]$
 $= -5 + x[2x^2 - 17x + 6]$

Then, $f(x) = 2x^3 - 17x^2 + 6x - 5$

Further, $f(2) = 42 = 2(2)^3 - 17(2)^2 + 6(2) - 5 = \boxed{-45}$

4) Given $u_{20} = 24.37$, $u_{22} = 49.28$, $u_{29} = 162.86$ & $u_{32} = 240.5$
 find u_{28} by Newton's divided difference formula.

Sol: The divided difference table is as follows.

x	$f(x) = u_x$	1 st DD	2 nd DD	3 rd DD
$x_0 = 20$	$f(x_0) = 24.37$	$f(x_0, x_1)$ $\frac{49.28 - 24.37}{22 - 20}$ $= 12.455$	$f(x_0, x_1, x_2)$ $\frac{162.86 - 49.28}{29 - 20} - \frac{16.226 - 12.455}{29 - 20}$ $= 0.419$	$f(x_0, x_1, x_2, x_3)$ $\frac{240.5 - 162.86}{32 - 29} - \frac{0.965 - 0.419}{32 - 29}$ $= 0.0455$
$x_1 = 22$	$f(x_1) = 49.28$	$f(x_1, x_2)$ $\frac{162.86 - 49.28}{29 - 22}$ $= 16.226$	$f(x_1, x_2, x_3)$ $\frac{240.5 - 162.86}{32 - 22} - \frac{25.88 - 16.226}{32 - 22}$ $= 0.965$	
$x_2 = 29$	$f(x_2) = 162.86$	$f(x_2, x_3)$ $\frac{240.5 - 162.86}{32 - 29}$ $= 25.88$		
$x_3 = 32$	$f(x_3) = 240.5$			

we have Newton's divided difference formula

$$f(x) = f(x_0) + (x-x_0)f'(x_0, x_1) + (x-x_0)(x-x_1)f(x_0, x_1, x_2) + (x-x_0)(x-x_1)(x-x_2)f(x_0, x_1, x_2, x_3) + \dots$$

$$\therefore f(28) = 24.37 + (28-20)(12.455) + (28-20)(28-22)(0.419) + (28-20)(28-22)(28-29)(0.0455)$$

$$\therefore f(28) = u_{28} = 141.938 \approx 141.94$$

5) Using Newton's divided difference formula find $f(8)$, $f(15)$ from the following data.

x	4	5	7	10	11	13
y	48	100	294	900	1210	2028

Sol: we have Newton's Divided difference formula.

$$f(x) = f(x_0) + (x-x_0)f(x_0, x_1) + (x-x_0)(x-x_1)f(x_0, x_1, x_2) + (x-x_0)(x-x_1)(x-x_2)f(x_0, x_1, x_2, x_3) + \dots$$

x	$f(x)$	1 st DD	2 nd DD	3 rd DD	4 th DD
$x_0 = 4$	$f(x_0) = 48$	$f(x_0, x_1)$ $\frac{100-48}{5-4} = 52$	$f(x_0, x_1, x_2)$ $\frac{97-52}{7-4} = 15$	$f(x_0, x_1, x_2, x_3)$ $= \frac{21-15}{10-4} = 1$	
$x_1 = 5$	$f(x_1) = 100$	$f(x_1, x_2)$ $\frac{294-100}{7-5} = 97$	$f(x_1, x_2, x_3)$ $= \frac{202-97}{10-5} = 21$	$f(x_1, x_2, x_3, x_4)$ $= \frac{27-21}{11-5} = 1$	0
$x_2 = 7$	$f(x_2) = 294$	$f(x_2, x_3)$ $= \frac{900-294}{10-7} = 202$	$f(x_2, x_3, x_4)$ $= \frac{310-202}{11-7} = 27$	$f(x_2, x_3, x_4, x_5)$ $= \frac{33-27}{13-7} = 1$	0
$x_3 = 10$	$f(x_3) = 900$	$f(x_3, x_4)$ $= \frac{1210-900}{11-10} = 310$	$f(x_3, x_4, x_5)$ $= \frac{409-310}{13-10} = 33$		
$x_4 = 11$	$f(x_4) = 1210$	$f(x_4, x_5)$ $= \frac{2028-1210}{13-11} = 409$			
$x_5 = 13$	$f(x_5) = 2028$				

Since two values are to be found, we prefer to find the interpolating polynomial $f(x)$.

$$f(x) = 48 + (x-4)(52) + (x-4)(x-9)(15) + (x-4)(x-9)(x-7)(1)$$

$f(x) = x^3 - x^2$ on simplification.

$$\therefore f(8) = 8^3 - 8^2 = 448, \quad f(19) = 19^3 - 19^2 = 3150$$

06) Determine $f(x)$ as a polynomial in x for the following data using Newton's divided difference formula.

x	-4	-1	0	2	5
y	1245	33	5	9	1335

07) Given the values.

x	5	7	11	13	17
$f(x)$	150	392	1452	2366	5202

using divided difference formula for unequal intervals to find $f(9)$.

$$f(9) = 810$$

Lagrange's interpolation formula's $\Rightarrow$

If $y_0 = f(x_0)$, $y_1 = f(x_1)$, $y_2 = f(x_2)$, ..., $y_n = f(x_n)$ be a set of values of an unknown funⁿ $y = f(x)$ corresponding to the values of x ; $x_0, x_1, x_2, \dots, x_n$ not necessarily at equal intervals then.

$$y = f(x) = \frac{(x-x_1)(x-x_2)\dots(x-x_n)}{(x_0-x_1)(x_0-x_2)\dots(x_0-x_n)} y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)\dots(x-x_n)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)\dots(x_1-x_n)} y_1 + \frac{(x-x_0)(x-x_1)(x-x_3)\dots(x-x_n)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)\dots(x_2-x_n)} y_2 + \dots + \frac{(x-x_0)(x-x_1)\dots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\dots(x_n-x_{n-1})} y_n$$

17 Use Lagrange's interpolation formula to find $f(4)$ given

x	0	2	3	6
$f(x)$	-4	2	14	158

Solⁿ: Let, $x_0 = 0$, $x_1 = 2$, $x_2 = 3$, $x_3 = 6$ } $x = 4$
 $y_0 = -4$, $y_1 = 2$, $y_2 = 14$, $y_3 = 158$ } $y = ?$

we have Lagrange's interpolation formula for four given values.

$$y = f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} y_1 + \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} y_3$$

Substituting the values we have

$$f(4) = \frac{(4-2)(4-3)(4-6)(-4)}{(0-2)(0-3)(0-6)} + \frac{(4-0)(4-3)(4-6)2}{(2-0)(2-3)(2-6)} + \frac{(4-0)(4-2)(4-6)14}{(3-0)(3-2)(3-6)} + \frac{(4-0)(4-2)(4-3)158}{(6-0)(6-2)(6-3)}$$

$$\Rightarrow \frac{(2)(1)(-2)(-4)}{(-2)(-3)(-6)} + \frac{(4)(1)(-2)2}{(2)(-1)(-4)} + \frac{(4)(2)(-2)14}{(3)(1)(-3)} + \frac{(4)(2)(1)158}{(6)(4)(3)}$$

$$\Rightarrow \frac{-4}{9} - 2 + \frac{224}{9} + \frac{158}{9} = \frac{382}{9} - \frac{22}{9} = \frac{360}{9} = 40$$

Thus $f(4) = 40$

x	0	1	3	4
y	-12	0	6	12

Hence estimate y at $x=2$.

solⁿ: let $x_0=0$ $x_1=1$ $x_2=3$ $x_3=4$ } $y=f(x)=?$
 $y_0=-12$ $y_1=0$ $y_2=6$ $y_3=12$

we have Lagrange's interpolation formula.

$$y = f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)y_0}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} + \frac{(x-x_0)(x-x_2)(x-x_3)y_1}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} \\ + \frac{(x-x_0)(x-x_1)(x-x_3)y_2}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} + \frac{(x-x_0)(x-x_1)(x-x_2)y_3}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}$$

Now, $y = f(x) = \frac{(x-1)(x-3)(x-4)(-12)}{(-1)(-3)(-4)} + 0 +$

$$\frac{x(x-1)(x-4)6}{(3)(2)(-1)} + \frac{x(x-1)(x-3)12}{(4)(3)(1)}$$

$$f(x) = (x-1)(x-3)(x-4) - x(x-1)(x-4) + x(x-1)(x-3) \\ = (x-1)[(x^2-7x+12) - (x^2-4x) + (x^2-3x)] \\ = (x-1)[x^2-6x+12] = x^3-7x^2+18x-12$$

Thus the required polynomial is

$$f(x) = x^3 - 7x^2 + 18x - 12$$

Now, $f(2) = 2^3 - 7(2)^2 + 18(2) - 12 = 4$

Thus $f(2) = 4$

4> The following table gives the normal weights of babies during first eight months of life

Age (in months)	0	2	5	8
weight (in pounds)	6	10	12	16

Estimate the weight of the baby at the age of seven months using L.T.F.

$$f(7) = 13.97 \approx 14$$

5> Apply Lagrange's formula inversely to find a root of the Eqⁿ $f(x)=0$ given that $f(30)=-30$, $f(34)=-13$, $f(38)=3$, $f(42)=18$.

$$x(0) = 37.2303$$

- 6) The observed values of a funⁿ are respectively 168, 120, 72 & 63 at the four positions 3, 7, 9, 10 of the independent variable. what is the best estimate you can give for the value of the funⁿ at the position 6 of the independent variable.
 (or) $f(6) = 147$

Numerical integration :-

There are three rules / formulae to obtain the value of the definite integral $I = \int_a^b y \cdot dx$ numerically.

The interval $[a, b]$ is divided into n equal parts of width h where, $h = (b-a)/n$

Let $a = x_0, x_1 = x_0 + h, x_2 = x_0 + 2h, \dots, x_n = x_0 + nh = b$ be the points of division.

Also, let $y_0 = f(x_0), y_1 = f(x_1), \dots, y_n = f(x_n)$ be the corresponding values of $y = f(x)$

Now we have a set of values of $y = f(x)$ at equidistant points of x and the values (x, y) are tabulated.

x	$x_0 = a$	x_1	x_2	x_3	---	$x_n = b$
y	y_0	y_1	y_2	y_3		y_n

The rules are as follows.

- # Simpson's one third rule - (even intervals)

$$I = \frac{h}{3} [(y_0 + y_n) + 4(y_1 + y_3 + \dots + y_{n-1}) + 2(y_2 + y_4 + \dots + y_{n-2})]$$

3 first last 4 (odd) 2 (even)

- # Simpson's three eighth rule - (3 multiples)

$$I = \frac{3h}{8} [(y_0 + y_n) + 3(y_1 + y_2 + y_4 + y_5 + \dots + y_{n-1}) + 2(y_3 + y_6 + \dots + y_{n-3})]$$

8 first last 3 (non multiple of 3) 2 (multiple of 3)

- # Trapezoidal rule :-

$$\int_a^b f(x) dx = h \left(\frac{y_0 + y_n}{2} + y_1 + y_2 + \dots + y_{n-1} \right)$$

It is applicable on any no. of intervals

$$I = \frac{h}{2} [y_0 + 2(y_1 + y_2 + y_3 + \dots + y_{n-1}) + y_n]$$

Problem :

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1) Evaluate $\int_2^7 \frac{dx}{x}$ using Trapezoidal rule. taking $n=5$

sol: $x_0 = 2, x_0 + nh = 7, n = 5$
 $2 + 5h = 7$

$5h = 5 \Rightarrow h = 1$

x	2	3	4	5	6	7
$y = \frac{1}{x}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{6}$	$\frac{1}{7}$
	y_0	y_1	y_2	y_3	y_4	y_5

By Trapezoidal rule $\int_{x_0}^{x_0+nh} y dx = \frac{h}{2} [(y_0 + y_5) + 2(y_1 + y_2 + y_3 + y_4)]$

$\Rightarrow \int_2^7 \frac{dx}{x} = \frac{1}{2} \left[\left(\frac{1}{2} + \frac{1}{7} \right) + 2 \left(\frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} \right) \right]$

$\int_2^7 \frac{dx}{x} = 1.2714$

2) Use the Trapezoidal rule with 6 subintervals to approximate

$\int_0^2 \frac{dx}{16+x^2}$

$x_0 = 0, x_0 + nh = 2, n = 6$

$0 + 6h = 2 \Rightarrow h = 1/3$

x	0	$1/3$	$2/3$	1	$4/3$	$5/3$	2
$y = \frac{1}{16+x^2}$	0.0625	0.062	0.0608	0.0588	0.0562	0.0532	0.05
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

By Trapezoidal rule $\int_{x_0}^{x_0+nh} y dx = \frac{h}{2} [(y_0 + y_6) + 2(y_1 + y_2 + y_3 + y_4 + y_5)]$

$\Rightarrow \int_0^2 \frac{dx}{16+x^2} = \frac{1/3}{2} [(0.0625 + 0.05) + 2(0.062 + 0.0608) + 0.0588 + 0.0562 + 0.0532]$

$\int_0^2 \frac{dx}{16+x^2} = 0.1158$

3) Find the solⁿ using trapezoidal rule

x	1.4	1.6	1.8	2.0	2.2
$f(x)$	4.0552	4.9530	6.0436	7.3891	9.0250

$\int_{x_0}^{x_0+nh} y dx = 4.9852$

4) Evaluate $I = \int_0^1 \frac{1}{1+x^2} dx$ by Simpson's $\frac{1}{3}$ rd rule By considering 6 subintervals of the interval $[0,1]$ Hence find an appropriate value of π

Solⁿ: $x_0 = 0$, $x_0 + nh = 1 \Rightarrow 0 + 6h = 1$, $n = 6$

$$h = 1/6$$

x	0	1/6	2/6	3/6	4/6	5/6	6/6 = 1
$y = \frac{1}{1+x^2}$	1	0.9730	0.9	0.8	0.6923	0.5901	0.5
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

By Simpson's $\frac{1}{3}$ rd rule

$$\int_0^1 \frac{1}{1+x^2} dx = \frac{h}{3} [(y_0 + y_6) + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)]$$

$$\int_0^1 \frac{1}{1+x^2} dx = 0.7854$$

To find π : we have $\int_0^1 \frac{1}{1+x^2} dx = 0.7854$

$$\tan^{-1} x \Big|_0^1 = 0.7854 \Rightarrow \tan^{-1} 1 - \tan^{-1} 0 = 0.7854$$

$$\Rightarrow \tan^{-1}(1) = 0.7854$$

$$\Rightarrow \tan^{-1} \frac{\pi}{4} = 0.7854$$

$$\boxed{\pi = 3.1416}$$

5) Evaluate $I = \int_0^5 \frac{1}{4x+5} dx$ by Simpson's $(\frac{1}{3})$ rd rule by considering 10 subintervals of the interval. Hence find an appropriate value of $\log 5$

Solⁿ: $x_0 = 0$, $x_0 + nh = 5 \Rightarrow 0 + 10h = 5 \Rightarrow h = 1/2$, $n = 10$

x	0	1/2	2/2	3/2	4/2	5/2	6/2	7/2	8/2
y	0.2	0.1428	0.1111	0.0909	0.0769	0.0666	0.0588	0.0526	0.0476
							9/2	10/2	
							0.0434	0.04	

By Simpson's $(\frac{1}{3})$ rd rule

$$\int_0^5 \frac{1}{4x+5} dx = \frac{h}{3} [(y_0 + y_{10}) + 4(y_1 + y_3 + y_5 + y_7 + y_9) + 2(y_2 + y_4 + y_6 + y_8)]$$

$$\Rightarrow \int_0^5 \frac{1}{4x+5} dx = 0.4023$$

To find $\log 5$: $\int_0^5 \frac{1}{4x+5} dx = 0.4023$

$$\frac{1}{4} \log(4x+5) \Big|_0^5 = 0.4023$$

$$\frac{1}{4} [\log(25) - \log(5)] = 0.4023$$

$$\log(25/5) = 1.6092 \Rightarrow \log(5) = 1.6092$$

6. Evaluate $\int_0^{\pi/2} \cos x \cdot dx$ by applying Simpson's $(1/3)^{rd}$ rule by taking eleven ordinates

solⁿ: $x_0 = 0, x_0 + nh = \pi/2 \Rightarrow 0 + 10h = \pi/2 \Rightarrow h = \pi/20 = 9^\circ$

x	0°	9°	18°	27°	36°	45°	54°	63°
$y = \cos x$	1	0.9877	0.9511	0.8910	0.8090	0.7071	0.5878	0.454
	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7
						72°	81°	90°
						0.3090	0.1564	0
						y_8	y_9	y_{10}

By Simpson's $(1/3)^{rd}$ rule

$$\int_a^b y \, dx = \frac{h}{3} [(y_0 + y_{10}) + 4(y_1 + y_3 + y_5 + y_7 + y_9) + 2(y_2 + y_4 + y_6 + y_8)]$$

$$\therefore \int_0^{\pi/2} \cos x \cdot dx = 1$$

Here $h = \frac{\pi}{20} \Rightarrow \pi = \frac{22}{7}$

$$1 + 4(3.1962) + 2(2.6635) = 1 + 12.7848 + 5.327$$

7. Using Simpson's $3/8^{th}$ rule evaluate $I = \int_0^1 \frac{x}{1+x^2} \, dx$ with $n=6$

Hence find an approximate value of i) $\log 2$ ii) $\log \sqrt{2}$

solⁿ: $x_0 = 0, x_0 + nh = 1 \Rightarrow 6h = 1 \Rightarrow h = 1/6, n=6$

x	0	$1/6$	$2/6$	$3/6$	$4/6$	$5/6$	$6/6 = 1$
$y = \frac{x}{1+x^2}$	0	0.1621	0.3	0.4	0.4615	0.4918	0.5
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

By Simpson's $(3/8)^{th}$ rule

$$\int_a^b y \, dx = \int_0^1 \frac{x}{1+x^2} \, dx = \frac{3h}{8} [(y_0 + y_6) + 3(y_1 + y_2 + y_4 + y_5) + 2(y_3)]$$

$$\int_0^1 \frac{x}{1+x^2} \, dx = 0.3467$$

To find $\log 2 \Rightarrow \int_0^1 \frac{x}{1+x^2} \, dx = 0.3467$

$$\frac{1}{2} \int_0^1 \frac{2x}{1+x^2} \, dx = 0.3467$$

$$\frac{1}{2} \log (1+x^2) \Big|_0^1 = 0.3467$$

$$\log 2 - \log 1 = 0.6934$$

$$\log 2 = 0.6934$$

To find $\log \sqrt{2}$:

$$\int_0^1 \frac{x}{1+x^2} dx = 0.3467 \quad ; \quad \frac{1}{2} \int_0^1 \frac{2x}{1+x^2} dx = 0.3467$$

$$\frac{1}{2} (\log 2 - \log 1) = 0.3467$$

$$\log \sqrt{2} = 0.3467$$

8> Use Simpson's 1/3rd rule find $\int_0^{0.6} e^{-x^2} dx$ by taking 6 sub intervals

soln: $\int_0^{0.6} e^{-x^2} dx = 0.5351$

$$h = 0.1$$