Numerical Methods - 2

Numerical solⁿ of ODEs :⇒

Numerical solⁿ of ODEs of first order and first degree.
Taylor's series method, Modified Euler's method, Runge-Kutta method of fourth order and Milne's predictor corrector formula.

Numerical methods for initial value problems :⇒

Consider a differential Eqⁿ of first order and first degree in the form $\frac{dy}{dx} = f(x,y)$ with the initial condition $y(x_0) = y_0$,

i.e $y = y_0$ at $x = x_0$. This problem of finding y is called initial value problem. We discuss numerical methods for solving an initial value problem.

In these numerical methods, we compute $y(x)$ in the neighbourhood of the value of x_0.

Equivalently, we compute $y(x_0 \pm h)$ where h is small enough. Lesser the value of h, results in greater the accuracy of $y(x_0 \pm h)$

Taylor's series method :⇒

consider the initial value problem $\frac{dy}{dx} = f(x,y)$ and $y(x_0) = y_0$. The solution $y(x)$ is approximated to a power series in $(x - x_0)$ using Taylor's series. Then we can find the value of y for various values of x in the neighbourhood of x_0.

We have Taylor's series expansion $y(x)$ about the point x_0 in the form:

$$y(x) = y(x_0) + (x-x_0)\cdot y'(x_0) + \frac{(x-x_0)^2}{2!} y''(x_0) + \frac{(x-x_0)^3}{3!} y'''(x_0) + \cdots$$

Here $y'(x_0)$, $y''(x_0)$ --- denote the value of the derivatives $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$ at x_0 which can be found by making use of the data.

1> a> Use Taylor's series method to find y at $x = 0.1, 0.2, 0.3$ considering terms upto the third degree given that $\frac{dy}{dx} = x^2 + y^2$ and $y(0) = 1$

b> compute $y(0.1)$

Sol^n:- Taylor's series expansion of $y(x)$ is given by

$$y(x) = y(x_0) + (x - x_0) \cdot y'(x_0) + \frac{(x-x_0)^2}{2!} y''(x_0) + \frac{(x-x_0)^3}{3!} y'''(x_0) + \cdots$$

By data, $x_0 = 0$, $y_0 = 1$ and $y' = x^2 + y^2$

$$\therefore y(x) = y(0) + x y'(0) + \frac{x^2}{2!} y''(0) + \frac{x^3}{3!} y'''(0) + \cdots \quad (1)$$

we need to compute $y'(0), y''(0), y'''(0)$

consider, $y' = x^2 + y^2$, $y'(0) = 0^2 + [y(0)]^2 = 0 + 1 = 1$

Diff w.r.to x

$y'' = 2x + 2yy'$: $y''(0) = (2)(0) + 2 \cdot y(0) \cdot y'(0)$
$= 2(1)(1) = 2$

Again Diff w.r.to x

$y''' = 2 + 2[yy'' + (y')^2]$: $y'''(0) = 2 + 2[(1)(2) + 1^2] = 8$

Substituting these values in (1) we have,

$$y(x) = 1 + x \cdot 1 + \frac{x^2}{2} \cdot 2 + \frac{x^3}{6} \cdot 8 = 1 + x + x^2 + \frac{4x^3}{3}$$

This is called as Taylor's series approximation upto the third degree and we need to put $x = 0.1, 0.2, 0.3$ in the same. Thus we have.

$$y(0.1) = 1 + 0.1 + (0.1)^2 + \frac{4(0.1)^3}{3} = 1.1113$$ (sol^n of b ends here)

$$y(0.2) = 1 + 0.2 + (0.2)^2 + \frac{4(0.2)^3}{3} = 1.2507$$

$$y(0.3) = 1 + 0.3 + (0.3)^2 + \frac{4(0.3)^3}{3} = 1.426$$

Thus, $y(0.1) = 1.1113$, $y(0.2) = 1.2507$, $y(0.3) = 1.426$

2> Find y at $x = 1.02$ correct to five decimal places given $dy = (xy - 1)\,dx$ and $y = 2$ at $x = 1$ applying Taylor's series method.

sol^n: Taylor's series expansion is given by

$$y(x) = y(x_0) + (x-x_0)y'(x_0) + \frac{(x-x_0)^2}{2!}y''(x_0) + \frac{(x-x_0)^3}{3!}y'''(x_0) + \cdots$$

By data, $x_0 = 1$, $y_0 = 2$ and $y' = \frac{dy}{dx} = xy - 1$

Since the number of derivatives for approximation is not specifically mentioned, we shall have the approximation upto 3rd degree

$$y(x) = y(1) + (x-1)y'(1) + \frac{(x-1)^2}{2!}y''(1) + \frac{(x-1)^3}{3!}y'''(1) \quad \text{——} (1)$$

consider, $y' = xy - 1 \quad : \quad y'(1) = (1)(2) - 1 = 1$

$y'' = xy' + y \quad : \quad y''(1) = (1)(1) + 2 = 3$

$y''' = xy'' + y' + y' \quad : \quad y'''(1) = (1)(3) + (1) + (1) = 5$

To find $y(1.02)$, we shall substitute these values along with $x = 1.02$ in (1)

$$y(1.02) = 2 + (1.02-1) + \frac{(1.02-1)^2}{2} \cdot 3 + \frac{(1.02-1)^3}{6} \cdot 5$$

$$= 2 + (0.02) + \frac{(0.02)^2(3)}{2} + \frac{(0.02)^3(5)}{6}$$

Thus, $y(1.02) = 2.02061$

3> Use Taylor's series method to find $y(4.1)$ given that $\frac{dy}{dx} = \frac{1}{x^2+y}$ and $y(4) = 4$

sol^n: Taylor's series expansion is given by

$$y(x) = y(x_0) + (x-x_0)y'(x_0) + \frac{(x-x_0)^2}{2!}y''(x_0) + \frac{(x-x_0)^3}{3!}y'''(x_0) + \cdots$$

By data, $y' = \frac{1}{x^2+y}$, $x_0 = 4$, $y_0 = 4$

$$\therefore y(x) = y(4) + (x-4)y'(4) + \frac{(x-4)^2}{2!}y''(4) + \frac{(x-4)^3}{3!}y'''(4) \quad \text{——} (1)$$

by approximating upto the third degree terms.

consider, $y' = \frac{1}{x^2+y}$ [Note: $y' = y'(x)$, $y'' = y''(x)$ etc]

or $y'(x^2+y) = 1$ —— (2)

substituting the initial values we have

$y'(4)[4^2+4] = 1$ or $y'(4) = \frac{1}{20} = 0.05$

Diff ② w.r.to x

$y'(2x+y') + (x^2+y) \cdot y'' = 0$ ——— ③

substituting the initial values and the value of $y'(4)$ we have, $0.05[(2)(4)+0.05] + [4^2+4]y''(4) = 0$

i.e $0.05[8+0.05] + 20y''(4) = 0$

i.e $0.4025 + 20y''(4) = 0$ or $y''(4) = -0.020125$

we observe that the value of the derivatives are small enough and the third degree term can also be neglected substituting these values in ① for computing $y(4.1)$ we have

$$y(4.1) = 4 + (4.1-4)(0.05) + \frac{(4.1-4)^2}{2}(-0.020125)$$

Thus, $y(4.1) = 4.0049$

4) Use Taylor's series method to solve $y' = x^2+y$ in the range $0 \le x \le 0.2$ by taking step size $h=0.1$ given that $y=10$ at $x=0$ initially considering terms upto the fourth degree.

Sol^n: In this problem, since the step size is specified as 0.1, the problem has to be done in two stages, we have to first find $y(0.1)$ and use this as the initial cond^n to compute $y(0.2)$

Taylor's series expansion is given by

$$y(x) = y(x_0) + (x-x_0)y'(x_0) + \frac{(x-x_0)^2}{2!}y''(x_0) + \frac{(x-x_0)^3}{3!}y'''(x_0) +$$

$$\frac{(x-x_0)^4}{4!}y^{(4)}(x_0) + \cdots \text{ ——— ①}$$

I^st stage: By data, $y' = x^2+y$, $x_0 = 0$, $y_0 = 10$

$y'(0) = 0^2 + 10 = 10$ or $y'(0) = 10$

Diff y' w.r.to x successively we have,

$y'' = 2x + y'$. $\quad y''(0) = (2)(0) + y'(0) = 0 + 10 = 10$

$y''' = 2 + y''$ $\quad y'''(0) = (2) + y''(0) = 2 + 10 = 12$

$y^4 = y'''$ $\quad y^{(4)}(0) = 12$

with $x = 0.1$ and $x_0 = 0$ ① becomes

$$y(0.1) = y(0) + (0.1)y'(0) + \frac{(0.1)^2}{2}y''(0) + \frac{(0.1)^3}{6}y'''(0) + \frac{(0.1)^4}{24}y^4(0)$$

$$= 10 + (0.1)10 + \frac{0.01}{2}(10) + \frac{-0.001}{6}(12) + \frac{0.0001}{24}(12)$$

Thus, $y(0.1) = 11.05205 \approx 11.052$

IInd stage. Now taking $x_0 = 0.1$, $y_0 = 11.052$, we have

$y' = x^2 + y$; $y'(0.1) = (0.1)^2 + 11.052 = 11.062$

$y'' = 2x + y'$; $y''(0.1) = 2(0.1) + 11.062 = 11.262$

$y''' = 2 + y''$; $y'''(0.1) = 2 + 11.262 = 13.262$

$y^{(4)} = y'''$; $y^{4}(0.1) = 13.262$

with $x = 0.2$ and $x_0 = 0.1$ ① becomes

$$y(0.2) = y(0.1) + (0.1)y'(0.1) + \frac{0.01}{2}y''(0.1) + \frac{0.001}{6}y'''(0.1)$$

$$+ \frac{0.0001}{24}y^{(4)}(0.1)$$

$$= 11.052 + (0.1)(11.062) + \frac{0.01}{2}(11.262) + \frac{0.001}{6}(13.262)$$

$$+ \frac{0.0001}{24}(13.262)$$

$y(0.2) = 12.216776 \approx 12.2168$

5) Employ Taylor's series method to find y at $x = 0.1$ & 0.2 correct to four places of decimal in step size of 0.1 given the linear differential eqn $\frac{dy}{dx} - 2y = 3e^x$ whose solution passes through the origin.

Also find $y(0.1)$ and $y(0.2)$ by analytical method.

Soln: By data $y' = 2y + 3e^x$ and $y(0) = 0$. i.e $x_0 = 0$, $y_0 = 0$

Taylor's Series expansion is given by:

$$y(x) = y(x_0) + (x - x_0)\cdot y'(x_0) + \frac{(x-x_0)^2}{2!}y''(x_0) + \cdots \quad ①$$

step ①: we shall compute $y(0.1)$

consider, $y' = 2y + 3e^x$: $y'(0) = 2(0) + 3e^0 = 3$

$y'' = 2y' + 3e^x$: $y''(0) = 2(3) + 3 = 9$

$y''' = 2y'' + 3e^x$: $y'''(0) = 2(9) + 3 = 21$

from ① we have.

$$y(0.1) = y(0) + (0.1)y'(0) + \frac{(0.1)^2}{2}y''(0) + \frac{(0.1)^3}{6}y'''(0)$$

by considering terms upto third degree. Further we have

$y(0.1) = 0 + (0.1)(3) + \frac{0.01}{2}(9) + \frac{0.001}{6}(21)$

Thus $y(0.1) = 0.3485$

step ② :→ we shall compute $y(0.2)$

consider, $y' = 2y + 3e^x$ and let $x_0 = 0.1$, $y_0 = 0.3485$

Now, $y'(0.1) = 2y(0.1) + 3 \cdot e^{0.1}$: $y'(0.1) = 4.0125$

$y'' = 2y' + 3e^x$

$y''(0.1) = 2(4.0125) + 3 \cdot e^{0.1}$: $y''(0.1) = 11.3405$

$y''' = 2y'' + 3e^x$: $y'''(0.1) = 2(11.3405) + 3 \cdot e^{0.1}$

$y'''(0.1) = 25.9965$

we have from ①

$y(0.2) = y(0.1) + (0.1)\, y'(0.1) + \frac{0.01}{2} y''(0.1) + \frac{0.001}{6} y'''(0.1)$

$\therefore y(0.2) = 0.3485 + (0.1)(4.0125) + \frac{0.01}{2}(11.3405) + \frac{0.001}{6}(25.996)$

Thus, $y(0.2) = 0.8108$

⇒ Solution by analytical method.

$\frac{dy}{dx} - 2y = 3e^x$ is of the form $\frac{dy}{dx} + Py = Q$

where $P = -2$, $Q = 3e^x$

sol^n : $y \cdot e^{\int P dx} = \int Q \cdot e^{\int P dx} \cdot dx + C$.

i.e $y \cdot e^{-2x} = \int 3e^x \cdot e^{-2x} \cdot dx + c$ or $ye^{-2x} = 3 \cdot \int e^{-x} dx + c$

i.e $y \cdot e^{-2x} = -3e^{-x} + c$

$y = -3e^x + c \cdot e^{2x}$ is the general sol^n

Applying the condition $y(0) = 0$, the general sol^n becomes. $0 = -3 + c$ or $c = 3$

Hence, $y = 3(e^{2x} - e^x)$ is the sol^n

Let us substitute $x = 0.1$ & 0.2

Then $y(0.1) = 0.3487$ and $y(0.2) = 0.8113$ by the analytical method.

6> Apply T.S.M to find $y(0.1)$ from $\frac{dy}{dx} = e^x - y^2$ with $y(0) = 1$

Sol^n: Given $y' = e^x - y^2$ $\quad x_0 = 0,\ y_0 = 1$

$y'(x_0) = y'(0) = e^0 - 1^2 = 0$

Diff w.r.to x

$y'' = e^x - 2y \cdot y'$ $\quad ; \quad y''(0) = e^0 - 2(1)(0) = 1$

$y''' = e^x - 2[y \cdot y'' + (y')^2]$ $\quad ; \quad y'''(0) = e^0 - 2[(1)(1) + 0]$

$y'''(0) = 1 - 2 = -1$

T.S.M

$$y(x) = y(x_0) + (x - x_0) \cdot y'(x_0) + \frac{(x - x_0)^2}{2!} \cdot y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0)$$

$$y(0.1) = 1 + (0.1) \cdot (0) + \frac{(0.1)^2}{2}(1) + \frac{(0.1)^3}{3!}(-1)$$

$$y(0.1) = 1 + 0 + \frac{0.01}{2} + \frac{0.001}{6}$$

6) $y(0.1) = 1 + 0.005 - 0.000166$

$y(0.1) = 1.0048$

7) Find $y(0.2)$ by considering the terms upto 4th degree given $\frac{dy}{dx} - 2y = 3e^x$ and $y(0) = 0$ by T.S.M.

Sol^n: Given $y' - 2y = 3e^x$ $\quad x_0 = 0,\ y_0 = 0$

$y' = 3e^x + 2y$ $\quad$ (Already solved)

Modified Euler's method :→

consider the initial value problem $\frac{dy}{dx} = f(x, y)$

$y(x_0) = y_0$

we need to find y at $x_1 = x_0 + h$

we first obtain $y(x_1) = y_1$ by applying Euler's formula and this value is regarded as the initial approximation for y_1 usually denoted by $y_1^{(0)}$ also called as the predicted value of y_1

Euler's formula is given by

$y_1^{(0)} = y_0 + h f(x_0, y_0)$

Since the accuracy is poor in this formula this value y_1 is successively improved to desired degree of accuracy by the following Modified Euler's formula.

where the successive approximations are denoted by $y_1^{(1)}$, $y_1^{(2)}$, $y_1^{(3)}$, ... etc.

$y_1^{(1)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(0)})]$

$y_1^{(2)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(1)})]$

$y_1^{(3)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(2)})]$

Each of the succeding approximation is better than the preceding ones, They are called corrected values. Euler's formula & modified Euler formula jointly are also called as Euler's predictor and corrector formulae

problems :⇒

1) Given $\frac{dy}{dx} = 1 + \frac{y}{x}$, $y = 2$ at $x = 1$. Find the approximate value y at $x = 1.4$ by taking step size $h = 0.2$ applying Modified Euler's method. Also find the value of y at $x = 1.2$ and 1.4 from the analytical sol^n of the Eq^n.

sol^n: The problem has to be worked in two stages.

Ist stage: $x_0 = 1$, $y_0 = 2$, $f(x, y) = 1 + (y/x)$, $h = 0.2$

$x_1 = x_0 + h = 1.2$, $y(x_1) = y_1 = y(1.2) = ?$

Now, $f(x_0, y_0) = 1 + (2/1) = 3$.

we have Euler's formula : $y_1^{(0)} = y_0 + hf(x_0, y_0)$ —— ①

$y_1^{(0)} = 2 + (0.2)(3) = 2.6$

Further we have modified Euler's formula :

$y_1^{(1)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(0)})]$ —— ②

$= 2 + (0.1)[3 + (1 + y_1^{(0)}/x_1)]$

$= 2 + (0.1)[4 + y_1^{(0)}/1.2]$

$= 2 + (0.1)[4 + 2.6/1.2]$

$\therefore y_1^{(1)} = 2.6167$

Next approximation $y_1^{(2)}$ is got just by replacing the value of $y_1^{(1)}$ in place of $y_1^{(0)}$

Now, $y_1^{(2)} = 2 + (0.1)[4 + 2.6167/1.2] = 2.6181$

Again, $y_1^{(3)} = 2 + (0.1)[4 + 2.6181/1.2] = 2.6182$

Also, $y_1^{(4)} = 2.6182$

Thus, $y(1.2) = 2.6182$

IInd stage : We repeat the process by taking $y(1.2) = 2.6182$ as the initial condition.

$x_0 = 1.2, \; y_0 = 2.6182 \; : \; f(x_0, y_0) = 1 + (y_0/x_0) = 3.1818$

$x_1 = x_0 + h = 1.4, \; y(x_1) = y_1 = y(1.4) = ?$

we have from (1)

$y_1^{(0)} = 2.6182 + (0.2)(3.1818) = 3.2546$

Now from (2).

$y_1^{(1)} = 2.6182 + (0.1)[3.1818 + (1 + y_1^{(0)}/x_1)]$

i.e $y_1^{(1)} = 2.6182 + (0.1)[4.1818 + 3.2546/1.4] = 3.2689$

$y_1^{(2)} = 2.6182 + (0.1)[4.1818 + 3.2689/1.4] = 3.2699$

$y_1^{(3)} = 2.6182 + (0.1)[4.1818 + 3.2699/1.4] = 3.2699$

Thus $y(1.4) = 3.2699 \cong 3.27$

Now, let us find the analytical sol^n of the eq^n.

$$\frac{dy}{dx} = 1 + \frac{y}{x} \quad \text{or} \quad \frac{dy}{dx} - \frac{y}{x} = 1$$

This is a linear DE of the form $\frac{dy}{dx} + Py = Q$ whose solution is given by. $y \cdot e^{\int P dx} = \int Q \cdot e^{\int P dx} \cdot dx + C$

Here, $P = \frac{-1}{x}$ and $Q = 1$

$\therefore e^{\int P dx} = e^{-\int 1/x \, dx} = e^{-\log x} = 1/x$

The sol^n becomes, $y \cdot 1/x = \int 1 \cdot 1/x \cdot dx + c$

i.e $y/x = \log x + c$

Applying the initial cond^n that $y = 2$ and $x = 1$ we have

$2/1 = \log 1 + c \quad \therefore c = 2$ since $\log(1) = 0$

The sol^n now becomes.

$y/x = \log x + 2$ or $y = x(\log x + 2)$

This is the analytical sol^n of the given initial value problem

Now by putting $x = 1.2$ and 1.4 we obtain,

$y(1.2) = 1.2(\log_e 1.2 + 2) = 2.6188$

$y(1.4) = 1.4(\log_e 1.4 + 2) = 3.2711$

Q2) Using M.E.M find y at $x = 0.2$ given $\frac{dy}{dx} = 3x + \frac{1}{2}y$ with $y(0) = 1$ taking $h = 0.1$ perform three iterations at each step

Sol?: we need to find $y(0.2)$ by taking $h = 0.1$

In two stages.

I stage :→ By data $x_0 = 0$, $y_0 = 1$, $h = 0.1$, $f(x, y) = 3x + (y/2)$

$f(x_0, y_0) = 0.5$, $x_1 = x_0 + h = 0.1$

$y(x_1) = y_1 = y(0.1) = ?$

From Euler's formula : $y_1^{(0)} = y_0 + h f(x_0, y_0)$ we obtain

$y_1^{(0)} = 1 + (0.1)(0.5) = 1.05$

we have modified Euler's formula

$$y_1^{(1)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(0)})]$$

$$= 1 + \frac{0.1}{2}\left[0.5 + 3x_1 + \frac{y_1^{(0)}}{2}\right]$$

$$= 1 + 0.05\left[0.5 + 3(0.1) + \frac{y_1^{(0)}}{2}\right]$$

$$y_1^{(1)} = 1 + 0.05\left[0.8 + \frac{y_1^{(0)}}{2}\right]$$

$$y_1^{(1)} = 1 + 0.05\left[0.8 + \frac{1.05}{2}\right] = 1.06625$$

The 2nd iterative value is got simply by replacing $y_1^{(0)}$ by $y_1^{(1)}$ ie by replacing 1.06625 inplace of 1.05

$$y_1^{(2)} = 1 + 0.05\left[0.8 + \frac{1.06625}{2}\right] = 1.0667$$

$$y_1^{(3)} = 1 + 0.05\left[0.8 + \frac{1.0667}{2}\right] = 1.0667$$

Thus $y(0.1) = 1.0667$

IInd stage : Now let $x_0 = 0.1$, $y_0 = 1.0667$

$f(x, y) = 3x + (y/2)$

$$f(x_0, y_0) = 3(0.1) + \frac{1.0667}{2} = 0.83335$$

$y_1^{(1)} = 1.1671$ $\quad y_1^{(2)} = 1.1675$ $\quad y_1^{(3)} = 1.1676$

$y(0.2) = 1.1676$

3) Using modified Euler's method find $y(0.2)$ correct to four decimal places solving the eq^n $\frac{dy}{dx} = x - y^2$, $y(0) = 1$ taking $h = 0.1$

compute $y(0.1)$ taking $h = 0.1$

sol^n:- we shall first compute $y(0.1)$ and use this value to compute $y(0.2)$

I stage :- By data $x_0 = 0$, $y_0 = 1$, $h = 0.1$, $f(x, y) = x - y^2$

$f(x_0, y_0) = 0 - 1^2 = -1$, $x_1 = x_0 + h = 0.1$

$y(x_1) = y_1 = y(0.1) = ?$

From Euler's formula : $y_1^{(0)} = y_0 + h f(x_0, y_0)$ we obtain

$y_1^{(0)} = 1 + (0.1)(-1) = 0.9$

we have modified Euler's formula

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(0)})\right]$$

$$= 1 + \frac{0.1}{2}\left[-1 + x_1 - \{y_1^{(0)}\}^2\right]$$

$$= 1 + 0.05\left[-1 + 0.1 - (0.9)^2\right]$$

$$= 1 + 0.05\left[-0.9 - (0.9)^2\right] = 0.9145$$

$y_1^{(2)} = 1 + 0.05\left[-0.9 - (0.9145)^2\right] = 0.9132$

$y_1^{(3)} = 0.9133$

Thus $y(0.1) = 0.9133$

II stage :- Now, let $x_0 = 0.1$, $y_0 = 0.9133$, $f(x, y) = x - y^2$

$f(x_0, y_0) = 0.1 - (0.9133)^2 = -0.7341$

$x_1 = x_0 + h = 0.2$, $y(x_1) = y(0.2) = ?$

Substituting in the Euler's formula

$y_1^{(0)} = 0.9133 + (0.1)(-0.7341) = 0.8399$

Now from modified Euler's formula.

$$y_1^{(1)} = 0.9133 + \frac{0.1}{2}\left[-0.7341 + x_1 - \{y_1^{(0)}\}^2\right]$$

$$y_1^{(1)} = 0.9133 + 0.05\left[-0.7341 + 0.2 - (0.8399)^2\right]$$

$$= 0.9133 + 0.05\left[-0.5341 - (0.8399)^2\right] = 0.8513$$

$y_1^{(2)} = 0.9133 + 0.05\left[-0.5341 - (0.8513)^2\right] = 0.8504$

$y_1^{(3)} = 0.9133 + 0.05\left[-0.5341 - (0.8504)^2\right] = 0.8504$

Thus $y(0.2) = 0.8504$

4) Use modified Euler's method to compute $y(0.1)$ given that $\frac{dy}{dx} = x^2+y$, $y(0)=1$ by taking $h=0.05$ considering the accuracy upto two approximations in each step

Sol: We need to compute $y(0.05)$ first and use this value to compute $y(0.1)$

I stage: By data $x_0=0$, $y_0=1$, $f(x,y)=x^2+y$, $h=0.05$

$f(x_0,y_0) = 0^2+1=1$, $x_1 = x_0+h = 0.05$

$y(x_1) = y_1 = y(0.05) = ?$

From Euler's formula: $y_1^{(0)} = y_0 + h f(x_0,y_0)$ we obtain

$y_1^{(0)} = 1+(0.05)(1) = 1.05$

Next by modified Euler's formula

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0,y_0)+f(x_1,y_1^{(0)})\right]$$

$$= 1+\frac{0.05}{2}\left[1+x_1^2+y_1^{(0)}\right]$$

$$= 1+0.025\left[1+(0.5)^2+1.05\right]$$

$$= 1+0.025[1.0025+1.05] = 1.0513$$

$$y_1^{(2)} = 1+0.025[1.0025+1.0513] = 1.0513$$

Thus, $y(0.05) = 1.0513$.

II stage: Now, let $x_0=0.05$, $y_0=1.0513$

$f(x,y)=x^2+y$; $f(x_0,y_0) = (0.05)^2+1.0513 = 1.0538$

$x_1 = x_0+h = 0.1$; $y(x_1) = y_1 = y(0.1) = ?$

Substituting in the Euler's formula

$y_1^{(0)} = 1.0513+0.05(1.0538) = 1.104$

Next from the modified Euler's formula.

$$y_1^{(1)} = 1.0513+\frac{0.05}{2}\left[1.0538+x_1^2+y_1^{(0)}\right]$$

$$= 1.0513+0.025\left[1.0538+(0.1)^2+1.104\right]$$

$$= 1.0513+0.025[1.0638+1.104] = 1.1055$$

$$y_1^{(2)} = 1.0513+0.025[1.0638+1.1055] = 1.1055$$

Thus the required $y(0.1) = 1.1055$

5) Using Euler's predictor and formulae solve $\frac{dy}{dx} = x+y$ at $x=0.2$ given that $y(0)=1$

Sol^n: we need to compute $y(0.2)$ and since the step size is not specified we shall take it be 0.2 itself.
By data we have, $x_0 = 0$, $y_0 = 1$, $f(x,y) = x+y$ & $h = 0.2$

$f(x_0, y_0) = 0+1 = 1$, $x_1 = x_0 + h = 0.2$

$y(x_1) = y_1 = y(0.2) = ?$

Since we have Eulers formula

$y_1^{(0)} = y_0 + h f(x_0, y_0) = 1 + (0.2)(1) = 1.2$

Next consider modified Eulers formula

$$y_1^{(1)} = y_0 + \frac{h}{2}[f(x_0,y_0) + f(x_1, y_1^{(0)})]$$

$$= 1 + \frac{0.2}{2}[1 + x_1 + y_1^{(0)}]$$

$$= 1 + 0.1[1 + 0.2 + (1.2)] = 1.24$$

$y_1^{(2)} = 1 + 0.1[1.2 + (1.24)] = 1.244$

$y_1^{(3)} = 1 + 0.1[1.2 + 1.244] = 1.2444$

$y_1^{(4)} = 1 + 0.1[1.2 + 1.2444] = 1.24444$

Thus the required $y(0.2) = 1.2444$

67 Using Euler's predictor-corrector formula compute $y(1.1)$ correct to five decimal places given that $\frac{dy}{dx} + \frac{y}{x} = \frac{1}{x^2}$ and $y=1$ at $x=1$

Sol^n: By data $\frac{dy}{dx} = \frac{1}{x^2} - \frac{y}{x}$ or $\frac{dy}{dx} = \frac{1-xy}{x^2}$

$x_0 = 1$, $y_0 = 1$ let us take $h = 0.1$

$f(x_0, y_0) = 0$, $x_1 = x_0 + h = 1.1$

$y(x_1) = y_1 = y(1.1) = ?$

$y(1.1) = 0.99605$.

Runge-Kutta method of fourth order $B \Rightarrow$

Consider the initial value problem $\frac{dy}{dx} = f(x,y)$, $y(x_0) = y_0$

we need to find $y(x_0+h)$ where h is the step size we have to first compute K_1, K_2, K_3, K_4 by the following formulae.

$K_1 = hf(x_0, y_0)$

$K_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}\right)$

$K_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}\right)$

$K_4 = hf(x_0 + h, y_0 + K_3)$

The required $y(x_0+h) = y_0 + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$

Problems :-

1) Given $\frac{dy}{dx} = 3x + \frac{y}{2}$, $y(0) = 1$, compute $y(0.2)$ by taking $h = 0.2$ using Runge-kutta method of fourth order. Also find the analytical solution.

sol^n: By data, $f(x,y) = 3x + \frac{y}{2}$, $x_0 = 0$, $y_0 = 1$, $h = 0.2$

we shall first compute K_1, K_2, K_3, K_4

$K_1 = hf(x_0, y_0) = (0.2) f(0,1) = (0.2)\left[3 \times 0 + \frac{1}{2}\right] = 0.1$

$K_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}\right) = (0.2) f\left(0 + \frac{0.2}{2}, 1 + \frac{0.1}{2}\right)$

$K_2 = (0.2) f(0.1, 1.05) = (0.2)\left[3 \times 0.1 + \frac{1.05}{2}\right] = 0.165$

$K_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}\right) = (0.2) f\left(0 + \frac{0.2}{2}, 1 + \frac{0.165}{2}\right)$

$K_3 = (0.2) f(0.1, 1.0825) = (0.2)\left[3 \times 0.1 + \frac{1.0825}{2}\right] = 0.16825$

$K_4 = hf(x_0 + h, y_0 + K_3) = (0.2) f(0.2, 1.16825)$

$K_4 = (0.2)\left[3 \times 0.2 + \frac{1.16825}{2}\right] = 0.236825$

we have, $y(x_0 + h) = y_0 + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$

$y(0.2) = 1 + \frac{1}{6}(0.1 + 2 \times 0.165 + 2 \times 0.16825 + 0.236825)$

Thus $y(0.2) = 1.1672208 \approx 1.1672$

we shall find the analytical sol^n of the given eq^n by writing in the form $\frac{dy}{dx} + Py = Q$ where sol^n is $y\, e^{\int P dx} = \int Q \cdot e^{\int P dx} dx + C$

we have $\frac{dy}{dx} - \frac{y}{2} = 3x$. Here $P = -\frac{1}{2}$, $Q = 3x$, $e^{\int P dx} = e^{-x/2}$

The sol^n becomes, $y \cdot e^{-x/2} = 3 \int x \cdot e^{-x/2} dx + C$

$y \cdot e^{-x/2} = 3 \left[x \cdot e^{-x/2}(-2) - \int e^{-x/2}(-2) \cdot 1\, dx \right] + C$

$y \cdot e^{-x/2} = 3\left[-2x e^{-x/2} - 4 \cdot e^{-x/2}\right] + C$

multiplying with $e^{x/2}$ we obtain, $y = -6x - 12 + c \cdot e^{x/2}$

Applying the initial cond^n that $y = 1$ at $x = 0$ the sol^n becomes

$1 = 0 - 12 + C$ $\quad \therefore C = 13$

The analytical sol^n of the initial value problem is given by

$y = -6x - 12 + 13\, e^{x/2}$

Now by putting $x = 0.2$ we have

$y(0.2) = -6(0.2) - 12 + 13\, e^{0.1} = 1.1672219$

Thus, $y(0.2) = 1.1672219 \simeq 1.1672$ by analytical method.

2> Use fourth order Runge-Kutta method to solve $(x+y)\frac{dy}{dx} = 1$, $y(0.4) = 1$ at $x = 0.5$ correct to 4 decimal places.

sol^n: we have, $\frac{dy}{dx} = \frac{1}{x+y}$ and $y = 1$ at $x = 0.4$

$f(x,y) = \frac{1}{x+y}$, $x_0 = 0.4$, $y_0 = 1$, $y(0.5) = ?$

Here, $x_0 + h = 0.5$, $h = 0.5 - x_0 = 0.5 - 0.4 = 0.1$

we shall first compute K_1, K_2, K_3, K_4

$K_1 = h\, f(x_0, y_0) = (0.1)\, f(0.4, 1) = (0.1)\left[\frac{1}{0.4+1}\right] = 0.0714$

$K_2 = h\, f\left(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}\right) = (0.1)\, f\left(0.4 + \frac{0.1}{2}, 1 + \frac{0.0714}{2}\right)$

$K_2 = (0.1)\, f(0.45, 1.0357) = (0.1)\left[\frac{1}{0.45 + 1.0357}\right] = 0.0673$

$K_3 = h f\left(x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}\right) = (0.1)\, f(0.45, 1.03365)$

$K_3 = (0.1)\left[\frac{1}{0.45 + 1.03365}\right] \simeq 0.0674$

$K_4 = hf(x_0+h, y_0+K_3) = (0.1)\, f(0.5, 1.0674)$

$K_4 = (0.1)\left[\dfrac{1}{0.5+1.0674}\right] = 0.0638$

we have, $y(x_0+h) = y_0 + \dfrac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$

$\therefore\ y(0.5) = 1 + \dfrac{1}{6}[0.0714 + 2(0.0673) + 2(0.0674) + 0.0638]$

$y(0.5) = 1.0674$

3) Using Runge-kutta method of fourth order, find $y(0.2)$ for the Eq^n $\dfrac{dy}{dx} = \dfrac{y-x}{y+x}$, $y(0) = 1$, taking $h = 0.2$

sol^n: By data $f(x,y) = \dfrac{y-x}{y+x}$, $x_0 = 0$, $y_0 = 1$, $h = 0.2$

we shall first compute K_1, K_2, K_3, K_4

$K_1 = hf(x_0, y_0) = (0.2)\, f(0,1) = (0.2)\left[\dfrac{1-0}{1+0}\right] = 0.2$

$K_2 = hf\left(x_0+\dfrac{h}{2}, y_0+\dfrac{K_1}{2}\right) = (0.2)\, f(0.1, 1.1)$

$K_2 = (0.2)\left[\dfrac{1.1-0.1}{1.1+0.1}\right] = 0.1667$

$K_3 = hf\left[x_0+\dfrac{h}{2}, y_0+\dfrac{K_2}{2}\right] = (0.2)\, f(0.1, 1.0835)$

$K_3 = (0.2)\left[\dfrac{1.0835-0.1}{1.0835+0.1}\right] = 0.1662$

$K_4 = hf(x_0+h, y_0+K_3) = (0.2)\, f(0.2, 1.1662)$

$K_4 = (0.2)\left[\dfrac{1.1662-0.2}{1.1662+0.2}\right] = 0.1414$

we have, $y_0(x_0+h) = y_0 + \dfrac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$

$\therefore\ y(0.2) = 1 + \dfrac{1}{6}[0.2 + 2(0.1667) + 2(0.1662) + 0.1414]$

$y(0.2) = 1.1679$

4> Use fourth order R.K method to find y at $x=0.1$ given that $\frac{dy}{dx} = 3e^x + 2y$, $y(0)=0$ and $h=0.1$

Sol^n: By data: $f(x,y) = 3e^x + 2y$, $x_0 = 0$, $y_0 = 0$, $h = 0.1$

we shall first compute k_1, k_2, k_3, k_4

$k_1 = h f(x_0, y_0) = (0.1) f(0,0) = (0.1)(3e^0 + 2\times 0) = 0.3$

$k_2 = h f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = (0.1) f(0.05, 0.15)$

$k_2 = (0.1)[3e^{0.05} + 2(0.15)] = 0.3454$

$k_3 = h f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = (0.1) f(0.05, 0.1727)$

$k_3 = (0.1)[3e^{0.05} + 2(0.1727)] = 0.3499$

$k_4 = h f(x_0 + h, y_0 + k_3) = 0.1 f(0.1, 0.3499)$

$k_4 = (0.1)[3e^{0.1} + 2(0.3499)] = 0.4015$

we have, $y(x_0 + h) = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$\therefore y(0.1) = 0 + \frac{1}{6}[0.3 + 2(0.3454) + 2(0.3499) + 0.4015]$

Thus $y(0.1) = 0.3487$

5> Use fourth order Runge-Kutta method to compute $y(1.1)$ given that $\frac{dy}{dx} = xy^{1/3}$, $y(1) = 1$

Sol^n: By data, $f(x,y) = xy^{1/3}$, $x_0 = 1$, $y_0 = 1$

we need to compute $y(1.1)$. This implies that $x_0 + h = 1.1$

$h = 0.1$

we shall first compute k_1, k_2, k_3, k_4

$k_1 = h f(x_0, y_0)$

$k_1 = (0.1) f(1, 1) = (0.1)[(1)(1)^{1/3}] = 0.1$

$k_2 = h f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = (0.1) f(1.05, 1.05)$

$k_2 = (0.1)[(1.05)(1.05)^{1/3}] = 1.067$

$k_3 = h f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = (0.1) f(1.05, 1.05335)$

$K_3 = (0.1)[(1.05)(1.05335)^{1/3}] = 0.1068$

$K_4 = hf(x_0+h, y_0+k_3) = (0.1) f(1.1, 1.1068)$

$K_4 = (0.1)[(1.1)(1.1068)^{1/3}] = 0.1138$

we have, $y(x_0+h) = y_0 + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$

$\therefore y(1.1) = 1 + \frac{1}{6}[0.1 + 2(0.1067) + 2(0.1068) + 0.1138]$

Thus $y(1.1) = 1.1068$

6) Using Runge-Kutta method of fourth order find $y(0.2)$ for the Eqn $\frac{dy}{dx} = \frac{y-x}{y+x}$, $y(0) = 1$, taking $h = 0.1$

compute $y(0.1)$

Soln:- The problem has to be done in 2 stages.

I stage: $f(x,y) = \frac{y-x}{y+x}$, $x_0 = 0$, $y_0 = 1$, $h = 0.1$

we shall first compute K_1, K_2, K_3, K_4

$K_1 = hf(x_0, y_0) = (0.1) f(0,1) = 0.1\left[\frac{1-0}{1+0}\right] = 0.1$

$K_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}\right) = (0.1) f(0.05, 1.05)$

$K_2 = (0.1)\left[\frac{1.05 - 0.05}{1.05 + 0.05}\right] = 0.091$

$K_3 = \cancel{to}\ hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}\right) = (0.1) f(0.05, 1.0455)$

$K_3 = (0.1)\left[\frac{1.0455 - 0.05}{1.0455 + 0.05}\right] = 0.0909$

$K_4 = hf(x_0+h, y_0+k_3) = (0.1) f(0.1, 1.0909)$

$K_4 = (0.1)\left[\frac{1.0909 - 0.1}{1.0909 + 0.1}\right] = 0.0832$

we have, $y(x_0+h) = y_0 + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$

$\therefore y(0.1) = 1 + \frac{1}{6}(0.1 + 0.182 + 0.1818 + 0.0832)$

Thus, $y(0.1) = 1.09167 \approx 1.0912$

IInd stage : $f(x,y) = \frac{y-x}{y+x}$, $x_0 = 0.1$, $y_0 = 1.0912$, $h = 0.1$

Again by using the same formulae for k_1, k_2, k_3, k_4 we have.

$k_1 = (0.1) f(0.1, 1.0912) = (0.1)\left[\frac{1.0912 - 0.1}{1.0912 + 0.1}\right] = 0.0832$

$k_2 = (0.1) f(0.15, 1.1328) = 0.0766$

$k_3 = (0.1) f(0.15, 1.1295) = 0.07655$

$k_4 = (0.1) f(0.2, 1.16775) = 0.07075$

Now, $y(x_0+h) = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$y(0.1+0.1) = 1.0912 + \frac{1}{6}(0.0832 + 0.1532 + 0.1531 + 0.07075)$

Thus, $y(0.2) = 1.167908 \approx 1.1679$

7) Solve $(y^2 - x^2)\, dx = (y^2 + x^2)\, dy$ for $x = 0.2$ & 0.4 given that $y = 1$ at $x = 0$ initially by applying R-K method of order four by taking $h = 0.2$

Sol^n: ~~k_1~~ Ist stage.

$k_1 = 0.2$

$k_2 = 0.1967$

$k_3 = 0.1967$

$k_4 = 0.1891$

$y(0.2) = 1.1959$

IInd stage.

$k_1 = 0.1891$

$k_2 = 0.1795$

$k_3 = 0.1793$

$k_4 = 0.1688$

$y(0.4) = 1.37525$.

Milne's predictor and corrector formulae :-

Consider the d.E. $\frac{dy}{dx}$ or $y' = f(x, y)$ with a set of four pre determined values of $y(x_0) = y_0$, $y(x_1) = y_1$, $y(x_2) = y_2$ and $y(x_3) = y_3$.
Also $x_4 = x_3 + h = x_0 + 4h$
predictor and corrector formulae to compute $y(x_4) = y_4$ are as follows.

$y_4^{(P)} = y_0 + \frac{4h}{3}(2y_1' - y_2' + 2y_3')$ (predictor formula)

$y_4^{(C)} = y_2 + \frac{h}{3}(y_2' + 4y_3' + y_4')$ (corrector formula)

Problems :-

1) Given that $\frac{dy}{dx} = x - y^2$ and the data $y(0) = 0$, $y(0.2) = 0.02$
$y(0.4) = 0.0795$, $y(0.6) = 0.1762$. compute y at $x = 0.8$ by applying Milne's method.

Sol^n:- we prepare the table using the given data which is essentially required for applying the predictor and corrector formula

x	y	$y' = x - y^2$
$x_0 = 0$	$y_0 = 0$	$y_0' = 0 - 0^2 = 0$
$x_1 = 0.2$	$y_1 = 0.02$	$y_1' = 0.2 - (0.02)^2 = 0.1996$
$x_2 = 0.4$	$y_2 = 0.0795$	$y_2' = 0.4 - (0.0795)^2 = 0.3937$
$x_3 = 0.6$	$y_3 = 0.1762$	$y_3' = 0.6 - (0.1762)^2 = 0.5689$
$x_4 = 0.8$	$y_4 = ?$	

By Milne's method (predictor formula)

$y_4^{(P)} = y_0 + \frac{4h}{3}(2y_1' - y_2' + 2y_3')$

$y_4^{(P)} = 0 + \frac{4(0.2)}{3}[2(0.1996) - 0.3937 + 2(0.5689)] = 0.3049$

Now, $y_4' = x_4 - y_4^2 = 0.8 - (0.3049)^2 = 0.707$

By corrector formula

$$y_4^{(c)} = y_2 + \frac{h}{3}(y_2' + 4y_3' + y_4')$$

$$y_4^{(c)} = 0.0795 + \frac{0.2}{3}[0.3937 + 4(0.5689) + 0.707] = 0.3046$$

Now, $y_4' = x_4 - y_4^2 = 0.8 - (0.3046)^2 = 0.7072$

Substituting this value of y_4' again in corrector formula

$$y_4^{(c)} = 0.0795 + \frac{0.2}{3}[0.3937 + 4(0.5689) + 0.7072] = 0.3046$$

Thus, $y_4 = y_4(0.8) = 0.3046$

2> Apply Milne's method to compute $y(1.4)$ correct to four decimal places given $\frac{dy}{dx} = x^2 + \frac{y}{2}$ and following data.

$y(1) = 2$, $y(1.1) = 2.2156$, $y(1.2) = 2.4649$, $y(1.3) = 2.7514$

Sol^n: First we shall prepare the following table

x	y	$y' = x^2 + y/2$
$x_0 = 1$	$y_0 = 2$	$y_0' = 1^2 + 2/2 = 2$
$x_1 = 1.1$	$y_1 = 2.2156$	$y_1' = (1.1)^2 + 2.2156/2 = 2.3178$
$x_2 = 1.2$	$y_2 = 2.4649$	$y_2' = (1.2)^2 + 2.4649/2 = 2.67245$
$x_3 = 1.3$	$y_3 = 2.7514$	$y_3' = (1.3)^2 + 2.7514/2 = 3.0657$
$x_4 = 1.4$		

we have, $y_4^{(P)} = y_0 + \frac{4h}{3}(2y_1' - y_2' + 2y_3')$

$$\therefore y_4^{(P)} = 2 + \frac{4(0.1)}{3}[2(2.3178) - 2.67245 + 2(3.0657)] = 3.0793$$

Hence, $y_4' = x_4^2 + \frac{y_4}{2} = (1.4)^2 + \frac{3.0793}{2} = 3.49965$

Now consider $y_4^{(c)} = y_2 + \frac{h}{3}(y_2' + 4y_3' + y_4')$

$$\therefore y_4^{(c)} = 2.4649 + \frac{0.1}{3}[2.67245 + 4(3.0657) + 3.49965] = 3.0794$$

Now $y_4^{(1)} = (1.4)^2 + \frac{3.0794}{2} = 3.4997$

$y_4 = y(1.4) = 3.0794$

3) The following table gives the sol^n of $5xy' + y^2 - 2 = 0$ find value of y at $x = 4.5$ using Milne's p-c formulae. Use the corrector formula twice.

x	4	4.1	4.2	4.3	4.4
y	1	1.0049	1.0097	1.0143	1.0187

Sol^n: By data $5xy' + y^2 - 2 = 0$ or $y' = \frac{2-y^2}{5x}$

x	y	$y' = \frac{2-y^2}{5x}$
$x_0 = 4$	$y_0 = 1$	$y_0' = \frac{2-1^2}{5\times4} = 0.05$
$x_1 = 4.1$	$y_1 = 1.0049$	$y_1' = \frac{2-(1.0049)^2}{5\times4.1} = 0.0483$
$x_2 = 4.2$	$y_2 = 1.0097$	$y_2' = \frac{2-(1.0097)^2}{5\times4.2} = 0.0467$
$x_3 = 4.3$	$y_3 = 1.0143$	$y_3' = \frac{2-(1.0143)^2}{5\times4.3} = 0.0452$
$x_4 = 4.4$	$y_4 = 1.0187$	$y_4' = \frac{2-(1.0187)^2}{5\times4.4} = 0.0437$
$x_5 = 4.5$	$y_5 = ?$	

we have milne's p-c formulae in the standard form

$$y_4^{(P)} = y_0 + \frac{4h}{3}(2y_1' - y_2' + 2y_3')$$

$$y_4^{(C)} = y_2 + \frac{h}{3}(y_2' + 4y_3' + y_4')$$

Since we require y_5, the Equivalent form of these formulae are given by

$$y_5^{(P)} = y_1 + \frac{4h}{3}(2y_2' - y_3' + 2y_4')$$

$$y_5^{(C)} = y_3 + \frac{h}{3}(y_3' + 4y_4' + y_5')$$

Hence, $y_5^{(P)} = 1.0049 + \frac{4(0.1)}{3}[2(0.0467) - 0.04527 + 2(0.0437)]$

$y_5^{(P)} = 1.023$

Now $y_5' = \frac{2 - y_5^2}{5x_5} = \frac{2-(1.023)^2}{5 \times 4.5} = 0.0424$

Hence, $y_5^{(C)} = 1.0143 + \frac{0.1}{3}[0.0452 + 4(0.0437) + 0.0424]$

$y_5^{(C)} = 1.023$

4) Find $y(0.8)$ from $\frac{dy}{dx} = x^3 + y$ given that $y(0) = 2$, $y(0.2) = 2.073$, $y(0.4) = 2.452$, $y(0.6) = 3.023$ by using Milne's predictor and corrector formula.

Soln:

x	y	$y' = x^3 + y$
0	2	$0 + 2 = 2 = y_0'$
0.2	2.073	$(0.2)^3 + 2.073 = 2.081 = y_1'$
0.4	2.452	$(0.4)^3 + 2.452 = 2.516 = y_2'$
0.6	3.023	$(0.6)^3 + 3.023 = 3.239 = y_3'$
0.8	?	

$y_4^{(P)} = y_0 + \frac{4h}{3}(2y_1' - y_2' + 2y_3') = 2 + \frac{4(0.2)}{3}[4.162 + 2.516 + 6.478]$

$y_4^{(P)} = 2 + 0.266[8.124] = 2 + 2.1609 = 4.16098$

$y_4' = x_4^3 + y_4 = (0.8)^3 + 4.16098 = 0.512 + 4.16098 = 4.6729$

$y_4^{(C)} = y_2 + \frac{h}{3}[y_2' + 4y_3' + y_4']$

$y_4^{(C)} = 2.452 + \frac{(0.2)}{3}[2.516 + 12.956 + 4.6729]$

$y_4^{(C)} = 2.452 + 0.0666(20.1449) = 2.452 + 1.3429$

$y_4^{(C)} = 3.7949$

5) If $\frac{dy}{dx} = 2e^x - y$, $y(0) = 2$, $y(0.1) = 2.010$, $y(0.2) = 2.040$ and $y(0.3) = 2.090$, find $y(0.4)$ correct to 4 decimal places by using Milne's predictor corrector method.

Sol^n:

x	y	$y' = 2e^x - y$
0	2	$y_0' = 2e^0 - 2 = 0$
0.1	2.010	$y_1' = 2e^{0.1} - 2.010 = 0.2003$
0.2	2.040	$y_2' = 2e^{0.2} - 2.04 = 0.4028$
0.3	0.090	$y_3' = 2e^{0.3} - 2.09 = 0.6097$
0.4		

By Milne's P-C method.

P-formula - $y_4^{(P)} = y_0 + \frac{4h}{3}[2y_1' - y_2' + 2y_3']$

$$y_4^{(P)} = 2 + \frac{4(0.1)}{3}[2(0.2003) - 0.4028 + 2(0.6097)]$$

$y_4^{(P)} = 2.1623$

Now $y_4' = 2e^{0.4} - 2.1623 = 0.8213$

C-formula - $y_4^{(C)} = y_2 + \frac{h}{3}[y_2' + 4y_3' + y_4']$

$$y_4^{(C)} = 2.04 + \frac{0.1}{3}[0.4028 + 4(0.6097) + 0.8213]$$

$y_4^{(C)} = 2.1621$

$y_4' = 2e^{0.4} - 2.1621 = 0.8215$

Applying corrector formula again

$$y_4^{(C)} = 2.04 + \frac{0.1}{3}[0.4028 + 4(0.6097) + 0.8215]$$

$y_4^{(C)} = 2.1621$

$y(0.4) = 2.1621$

6) Apply Milne's method to find $y(0.8)$ given $\frac{dy}{dx} + xy^2 = 0$

x	0	0.2	0.4	0.6
y	2	1.9231	1.7214	1.4706

$\frac{dy}{dx} = -xy^2$

x	y	$y' = -xy^2$
0	2	$y_0' = -0 \times 2^2 = 0$
0.2	1.9231	$y_1' = -0.2 \times (1.9231)^2 = -0.73966$
0.4	1.7214	$y_2' = -0.4 \times (1.7214)^2 = -1.18528$
0.6	1.4706	$y_3' = -0.6 \times (1.4706)^2 = -1.29759$
0.8		

$y_4^{(P)} = y_0 + \frac{4h}{3}[2y_1' - y_2' + 2y_3']$

$y_4^{(P)} = 2 + \frac{4(0.2)}{3}[2(-0.73966) + 1.18528 + 2(1.29759)]$

$[\overset{(+)}{-1.4793} \overset{(-)}{+} 1.018528 \overset{(+)}{-} 2.5951]$

$= 2 + 0.2666 - 2.88912$

$y_4^{(P)} = -0.62252$ $\qquad$ $y_4^{(P)} = 1.88912$

$y_4' = -(0.8)(0.62252)^2$

$y_4' = -0.31002$ $\qquad$ $y_4' = 2.8550$

$y_4^{(C)} = y_2 + \frac{h}{3}[y_2' + 4y_3' + y_4']$

$y_4^{(C)} = 1.7214 + \frac{(0.2)}{3}[\overset{(+)}{-1.18528} \overset{(+)}{-} 5.19036 \overset{(+)}{-} 0.31002]$

$1.7214 + 0.0666[-6.68566]$

$y_4^{(C)} = 1.7214 - 0.44526$ $\qquad$ $y_4^{(C)} = 2.1666$

$y_4^{(C)} = 1.27613$

7) find $y(2)$ by using Milne's P-C method given $\frac{dy}{dx} = \frac{x+y}{2}$ &

x	0	0.5	1	1.5
y	2	2.636	3.595	4.968

$y_4^{(P)} = 6.8661$

$y_4' = 4.43305$

$y_4^{(C)} = 6.8725$

$y_4' = 4.4362$

$y_4^{(C)} = 6.8720$