CALCULUS AND DIFFERENTIAL EQUATIONS (BMATS101)

Module 1 - Calculus

Prerequisites:

Trigonometry

Pythagorean identities

$$\sin^2\theta + \cos^2\theta = 1$$

$$\sec^2\theta - \tan^2\theta = 1$$

$$cosec^2\theta - \cot^2\theta = 1$$

Reciprocal ratios

$$\operatorname{cosec}\theta = \frac{1}{\sin\theta}$$

$$\sec\theta = \frac{1}{\cos\theta}$$

$$\cot\theta = \frac{1}{\tan\theta}$$

Sum formulas

$$\sin(x+y) = \sin x\cos y + \cos x\sin y$$

$$\cos(x+y) = \cos x\cos y - \cos x\cos y$$

$$\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x\tan y}$$

Difference formulas

$$\sin(x-y) = \sin x\cos y - \cos x\sin y$$

$$\cos(x-y) = \cos x\cos y + \cos x\cos y$$

$$\tan(x-y) = \frac{\tan x - \tan y}{1 + \tan x\tan y}$$

Double angle formulas

$$\sin 2x = 2\sin x\cos x$$

$$\cos 2x = \cos^2 x - \sin^2 x$$

$$\tan 2x = \frac{2\tan x}{1 - \tan^2 x}$$

Triple angle formulas

$$\sin 3x = 3\sin x - 4\sin^3 x$$

$$\cos 3x = 4\cos^3 x - 3\cos x$$

$$\tan\ 3x = \frac{3\tan x - \tan^3 x}{1 - 3\tan^2 x}$$

Half angle formulas

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$$

$$\cos^2 x = \frac{1}{2}(1 + \cos 2x)$$

$$\tan^2 x = \frac{1 - \cos 2x}{1 + \cos 2x}$$

Tangent formulas

$$\sin 2x = \frac{2\tan x}{1 + \tan^2 x}$$

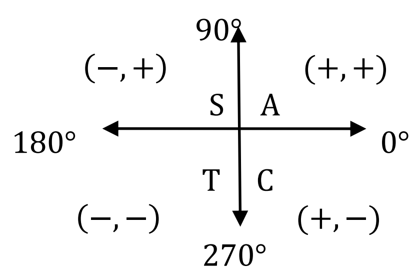
$$\cos 2x = \frac{1 - \tan^2 x}{1 + \tan^2 x}$$

$$\tan 2x = \frac{2\tan x}{1 - \tan^2 x}$$

Standard angle formulas

θ	$o°$	$30°$	$45°$	$60°$	$90°$
$\sin\theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞

ASTC Rule



Note:

$2\sin^2\frac{x}{2} = 1 - \cos x$	$\left(\cos\frac{x}{2} + \sin\frac{x}{2}\right)^2 = 1 + \sin x$	$\tan\left(\frac{\pi}{4} + x\right) = \frac{1 + \tan x}{1 - \tan x}$
$2\cos^2\frac{x}{2} = 1 + \cos x$	$\left(\cos\frac{x}{2} - \sin\frac{x}{2}\right)^2 = 1 - \sin x$	$\tan\left(\frac{\pi}{4} - x\right) = \frac{1 - \tan x}{1 + \tan x}$

Same ratio formulas:

$\sin(-\theta) = -\sin\theta$	$\sin(2\pi - \theta) = -\sin\theta$	$\sin(\pi - \theta) = \sin\theta$	$\sin(\pi + \theta) = -\sin\theta$
$\cos(-\theta) = \cos\theta$	$\cos(2\pi - \theta) = \cos\theta$	$\cos(\pi - \theta) = -\cos\theta$	$\cos(\pi + \theta) = -\cos\theta$
$\tan(-\theta) = -\tan\theta$	$\tan(2\pi - \theta) = -\tan\theta$	$\tan(\pi - \theta) = -\tan\theta$	$\tan(\pi + \theta) = \tan\theta$
$\cot(-\theta) = -\cot\theta$	$\cot(2\pi - \theta) = -\cot\theta$	$\cot(\pi - \theta) = -\cot\theta$	$\cot(\pi + \theta) = \cot\theta$
$\sec(-\theta) = \sec\theta$	$\sec(2\pi - \theta) = \sec\theta$	$\sec(\pi - \theta) = -\sec\theta$	$\sec(\pi + \theta) = -\sec\theta$
$\text{cosec}(-\theta) = -\text{cosec}\,\theta$	$\text{cosec}(2\pi - \theta) = -\text{cosec}\,\theta$	$\text{cosec}(\pi - \theta) = \text{cosec}\,\theta$	$\text{cosec}(\pi + \theta) = -\text{cosec}\,\theta$
(IV quadrant) Cos +ve	(IV quadrant) Cos +ve	(II quadrant) Sin +ve	(III quadrant) Tan +ve

Co ratio formulas:

$\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta$	$\sin\left(\frac{\pi}{2} + \theta\right) = \cos\theta$	$\sin\left(\frac{3\pi}{2} - \theta\right) = -\cos\theta$	$\sin\left(\frac{3\pi}{2} + \theta\right) = -\cos\theta$
$\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$	$\cos\left(\frac{\pi}{2} + \theta\right) = -\sin\theta$	$\cos\left(\frac{3\pi}{2} - \theta\right) = -\sin\theta$	$\cos\left(\frac{3\pi}{2} + \theta\right) = \sin\theta$
$\tan\left(\frac{\pi}{2} - \theta\right) = \cot\theta$	$\tan\left(\frac{\pi}{2} + \theta\right) = -\cot\theta$	$\tan\left(\frac{3\pi}{2} - \theta\right) = \cot\theta$	$\tan\left(\frac{3\pi}{2} + \theta\right) = -\cot\theta$
$\cot\left(\frac{\pi}{2} - \theta\right) = \tan\theta$	$\cot\left(\frac{\pi}{2} + \theta\right) = -\tan\theta$	$\cot\left(\frac{3\pi}{2} - \theta\right) = \tan\theta$	$\cot\left(\frac{3\pi}{2} + \theta\right) = -\tan\theta$
$\sec\left(\frac{\pi}{2} - \theta\right) = \text{cosec}\,\theta$	$\sec\left(\frac{\pi}{2} + \theta\right) = -co\sec\theta$	$\sec\left(\frac{3\pi}{2} - \theta\right) = -\text{cosec}\,\theta$	$\sec\left(\frac{3\pi}{2} + \theta\right) = co\sec\theta$
$\text{cosec}\left(\frac{\pi}{2} - \theta\right) = \sec\theta$	$\text{cosec}\left(\frac{\pi}{2} + \theta\right) = \sec\theta$	$\text{cosec}\left(\frac{3\pi}{2} - \theta\right) = -\sec\theta$	$\text{cosec}\left(\frac{3\pi}{2} + \theta\right) = -\sec\theta$
(I quadrant) All +ve	(II quadrant) Sin +ve	(III quadrant) Tan +ve	(IV quadrant) Cos +ve

Differentiation of some standard functions

Non Trigonometric functions	*Trigonometric functions*	*Hyperbolic functions*	*Inverse functions*
$(k)' = 0$	$(\sin x)' = \cos x$	$(\sinh x)' = \cosh x$	$(\sin^{-1}x)' = \frac{1}{\sqrt{1-x^2}}$
$(x^n)' = n\,x^{n-1}$	$(\cos x)' = -\sin x$	$(\cosh x)' = \sinh x$	$(\cos^{-1}x)' = -\frac{1}{\sqrt{1-x^2}}$
$(\sqrt{x})' = \frac{1}{2\sqrt{x}}$	$(\tan x)' = \sec^2 x$	$(\tanh x)' = \text{sech}^2 x$	$(\tan^{-1}x)' = \frac{1}{1+x^2}$
$(\log x)' = \frac{1}{x}$	$(\cot x)' = -\,\text{cosec}^2\,x$	$(\coth x)' = -\,\text{cosech}^2\,x$	$(\cot^{-1}x)' = -\frac{1}{1+x^2}$
$(e^x)' = e^x$	$(\sec x)' = \sec x.\tan x$	$(\text{sech}\,x)' = -\text{sech}\,x.\tanh x$	$(\sec^{-1}x)' = \frac{1}{x\sqrt{x^2-1}}$
$(a^x)' = a^x \log a$	$(\text{cosec}\,x)'$ $= -\text{cosec}\,x.\cot x$	$(\text{cosech}\,x)'$ $= -\text{cosech}\,x.\coth x$	$(\text{cosec}^{-1}x)' = -\frac{1}{x\sqrt{x^2-1}}$

Rules of differentiation

1. $(ku)' = ku'$ 2. $(u \pm v)' = u' \pm v'$	3. $(uv)' = uv' + vu'$ 4. $\left(\frac{u}{v}\right)' = \frac{vu'-uv'}{v^2}$

1.1 Polar curves

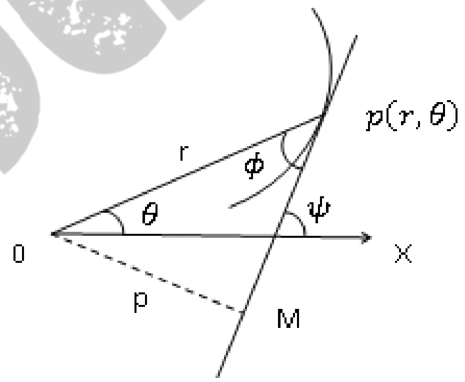
Introduction:

- Polar coordinates are $(x, y) = (r\cos\theta, r\sin\theta)$ where r - radial distance,θ- polar angle.
- Polar form of the equation of the curve $r = f(\theta)$ is called polar curve.
- $1 + \cos\theta = 2\cos^2\frac{\theta}{2}$, $\;1 - \cos\theta = 2\sin^2\frac{\theta}{2}$, $\;\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$
- $\tan\left(\frac{\pi}{4} + \theta\right) = \frac{1+\tan\theta}{1-\tan\theta}, \tan\left(\frac{\pi}{4} - \theta\right) = \frac{1-\tan\theta}{1+\tan\theta}$
- Angle between radius vector and tangent is $tan\phi = r\frac{d\theta}{dr}$

Problems:

1. **Derive angle between radius vector and tangent. (May 22)**

Let $P(r, \theta)$ be any point on the polar curve $r = f(\theta)$.



Let χ be the angle from the X axis to the tangent.

Let p be the perpendicular distance from the origin to the tangent.

By diagram, $\chi = \theta + \phi$

$\tan\chi = \tan(\theta + \phi)$

$\tan\chi = \frac{\tan\theta + \tan\phi}{1 - \tan\theta . \tan\phi}$ ------- (1)

But

$$\tan\chi = \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\frac{d}{d\theta}(r\sin\theta)}{\frac{d}{d\theta}(r\cos\theta)} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}$$

Divide by $\frac{dr}{d\theta}\cos\theta$ in numerator and denominator,

$$\tan\chi = \frac{\tan\theta + r\frac{d\theta}{dr}}{1 - r\frac{d\theta}{dr}\tan\theta} \quad \text{-------- (2)}$$

Equating components of (1) and (2),

$$\tan\phi = r\frac{d\theta}{dr}$$

2. Find the angle between radius vector and tangent to the following:
(i) $r^2\cos 2\theta = a^2$ (ii) $r = a(1+\cos\theta)$

(i) $r^2\cos 2\theta = a^2$ Take log on both sides, $\log(r^2\cos 2\theta) = \log a^2$ $\log r^2 + \log\cos 2\theta = 0$ $2\log r + \log\cos 2\theta = 0$ Differentiate with respect to θ, $\frac{2}{r}\frac{dr}{d\theta} = \frac{2\sin 2\theta}{\cos 2\theta}$ $\frac{1}{r}\frac{dr}{d\theta} = \tan 2\theta$ $\text{Cot}\,\phi = \cot\left(\frac{\pi}{2} - 2\theta\right)$ $\phi = \frac{\pi}{2} - 2\theta$	(ii) $r = a(1+\cos\theta)$ Take log on both sides, $\log r = \log a(1+\cos\theta)$ $\log r = \log a + \log(1+\cos\theta)$ Differentiate with respect to θ, $\frac{1}{r}\frac{dr}{d\theta} = 0 + \frac{-\sin\theta}{(1+\cos\theta)}$ $\frac{1}{r}\frac{dr}{d\theta} = \frac{-2\sin\frac{\theta}{2}.\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}}$ $\frac{1}{r}\frac{dr}{d\theta} = -\tan\frac{\theta}{2}$ $\cot\phi = \cot\left(\frac{\pi}{2} + \frac{\theta}{2}\right)$ At $\theta = \frac{\pi}{3}$, $\cot\phi = \cot\left(\frac{\pi}{2} + \frac{\pi}{6}\right)$ Angle between the radius vector and the tangent is $\phi = \frac{2\pi}{3}$

3. **Find the angle between radius vector and tangent to the following:**
(i) $r^n = a^n \sec(n\theta + \alpha)$ (ii) $r^m = a^m(\cos m\theta + \sin m\theta)$

(iii) $r^n = a^n \sec(n\theta + \alpha)$ Take log on both sides, $\log r^n = \log a^n + \log \sec(n\theta + \alpha)$ $n \log r = \log a^n + \log \sec(n\theta + \alpha)$ Differentiate with respect to θ, $\frac{n}{r}\frac{dr}{d\theta} = 0 + n\frac{\sec(n\theta+\alpha)\tan(n\theta+\alpha)}{\sec(n\theta+\alpha)}$ $\cot\phi = \tan(n\theta + \alpha)$ $\cot\phi = \cot\left(\frac{\pi}{2} - (n\theta + \alpha)\right)$ Angle between the radius vector and the tangent is $\phi = \frac{\pi}{2} - n\theta - \alpha$	(iv) $r^m = a^m(\cos m\theta + \sin m\theta)$ Take log on both sides, $\text{Log}\, r^m = \log a^m(\cos m\theta + \sin m\theta)$ $\log r^m = \log a^m + \log(\cos m\theta + \sin m\theta)$ Differentiate with respect to θ, $\frac{m}{r}\frac{dr}{d\theta} = \frac{m\,(\cos m\theta - \sin m\theta)}{(\cos m\theta + \sin m\theta)}$ $\text{Cot}\,\phi = \tan\left(\frac{\pi}{4} - m\theta\right)$ $\cot\phi = \cot\left(\frac{\pi}{2} - \left(\frac{\pi}{4} - m\theta\right)\right)$ Angle between the radius vector and the tangent is $\phi = \frac{\pi}{4} + m\theta$

4. **Find the angle between radius vector and tangent to the following:**
$\boldsymbol{\frac{l}{r} = 1 + e\cos\theta}$ **.**

$\frac{l}{r} = 1 + e\cos\theta$

Take log on both sides,

$\log l - \log r = \log(1 + e\cos\theta)$

Differentiate with respect to θ,

$$-\frac{1}{r}\frac{dr}{d\theta} = \frac{-e\sin\theta}{1+e\cos\theta}$$

$$\cot\phi = \frac{e\sin\theta}{1+e\cos\theta}$$

$$\tan\phi = \frac{1+e\cos\theta}{e\sin\theta}, \qquad \phi = \tan^{-1}\left(\frac{1+e\cos\theta}{e\sin\theta}\right)$$

5. **Show that the following pair of curves intersect orthogonally:**

(i) $\boldsymbol{r = a(1+\cos\theta)\,,\ r = b(1-\cos\theta)}$

$r = a(1+\cos\theta)$ Take log on both sides, $\log r = \log a + \log(1+\cos\theta)$ Differentiate w. r. to θ $\frac{1}{r}\frac{dr}{d\theta} = \frac{-\sin\theta}{1+\cos\theta}$ $\frac{1}{r}\frac{dr}{d\theta} = \frac{-2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}}$ $\cot\phi_1 = -\tan\frac{\theta}{2}$	$r = b(1-\cos\theta)$ Take log on both sides, $\log r = \log b + \log(1-\cos\theta)$ Differentiate w. r. to θ $\frac{1}{r}\frac{dr}{d\theta} = \frac{\sin\theta}{1-\cos\theta}$ $\frac{1}{r}\frac{dr}{d\theta} = \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\sin^2\frac{\theta}{2}}$ $\cot\phi_2 = \cot\frac{\theta}{2}$

Since $\cot\phi_1 . \cot\phi_2 = -1$, both intersect orthogonally.

(ii) $\boldsymbol{r^n = a^n\cos n\theta}$ **and** $\boldsymbol{r^n = b^n\sin n\theta}$

$r^n = a^n\cos n\theta$ Take log on both sides, n $\log r = n\log a + \log\cos n\theta$ Differentiate w. r. to θ $\frac{n}{r}\frac{dr}{d\theta} = \frac{-n\sin n\theta}{\cos n\theta} = -n\tan n\theta$ $\frac{1}{r}\frac{dr}{d\theta} = -\tan n\theta$ $\cot\phi_1 = -\tan n\theta$	$r^n = b^n\sin n\theta$ Take log on both sides, $\log r = n\log b + \log\sin n\theta$ Differentiate w. r. to θ $\frac{n}{r}\frac{dr}{d\theta} = \frac{n\sin n\theta}{\cos n\theta} = n\cot n\theta$ $\frac{1}{r}\frac{dr}{d\theta} = \cot n\theta$ $\cot\phi_2 = \cot n\theta$

Since $\cot\phi_1 . \cot\phi_2 = -1$, both intersect orthogonally.

(iii) $\boldsymbol{r = \frac{a}{1+\cos\theta}}$ **and** $\boldsymbol{r = \frac{b}{1-\cos\theta}}$

$r = \frac{a}{1+\cos\theta}$ Take log on both sides, $\log r = \log a - \log(1+\cos\theta)$ Differentiate w. r. to θ $\frac{1}{r}\frac{dr}{d\theta} = \frac{\sin\theta}{1+\cos\theta} = \tan\frac{\theta}{2}$ $\cot\phi_1 = \tan\frac{\theta}{2}$	$r = \frac{b}{1-\cos\theta}$ Take log on both sides, $\log r = \log b - \log(1-\cos\theta)$ Differentiate w. r. to θ $\frac{1}{r}\frac{dr}{d\theta} = -\frac{\sin\theta}{1-\cos\theta} = -\cot\frac{\theta}{2}$ $\cot\phi_2 = -\cot\frac{\theta}{2}$

Since $\cot\phi_1 . \cot\phi_2 = -1$, both intersect orthogonally

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(iv) $\boldsymbol{r = a\theta}$ **and** $\boldsymbol{r = \frac{a}{\theta}}$

$r = a\theta$ Take log on both sides, $\log r = \log a\theta$ $\log r = \log a + \log\theta$ Differentiate with respect to θ $\frac{1}{r}\frac{dr}{d\theta} = \frac{1}{\theta}$ $\cot\phi_1 = \frac{1}{\theta}$	$r = \frac{a}{\theta}$ Take log on both sides, $\log r = \log\frac{a}{\theta}$ $\log r = \log a - \log\theta$ Differentiate with respect to θ $\frac{1}{r}\frac{dr}{d\theta} = -\frac{1}{\theta}$ $\cot\phi_2 = -\frac{1}{\theta}$

But $a\theta = \frac{a}{\theta} \Rightarrow \theta^2 = 1$. Since $\cot\phi_1 . \cot\phi_2 = -\frac{1}{\theta^2} = -1$, both intersect orthogonally.

(v) $\boldsymbol{r = ae^{\theta}}$ **and** $\boldsymbol{re^{\theta} = b}$

$r = ae^{\theta}$ Take log on both sides, $\log r = \log a + \theta$ Differentiate with respect to θ $\frac{1}{r}\frac{dr}{d\theta} = 1$ $\cot\phi_1 = 1$	$re^{\theta} = a$ Take log on both sides, $\log r + \theta = \log a$ Differentiate with respect to θ $\frac{1}{r}\frac{dr}{d\theta} = -1$ $\cot\phi_2 = -1$

Since $\cot\phi_1 . \cot\phi_2 = -1$, both intersect orthogonally.

(vi) Show that $\boldsymbol{r = 4\sec^2\frac{\theta}{2}}$ $\boldsymbol{and\ r = 9\,cosec^2\frac{\theta}{2}}$ **the pair of curves cut orthogonally.**

(May 22)

$r = 4\sec^2\frac{\theta}{2}$ Take log on both sides, $\log r = \log 4 + 2\log\sec\frac{\theta}{2}$ Differentiate with respect to θ $\frac{1}{r}\frac{dr}{d\theta} = 0 + \frac{2}{\sec\frac{\theta}{2}}\sec\frac{\theta}{2}\tan\frac{\theta}{2}$ $\cot\phi_1 = 2\tan\frac{\theta}{2}$	$r = 9\,cosec^2\frac{\theta}{2}$ Take log on both sides, $\log r = \log 9 + 2\log cosec\frac{\theta}{2}$ Differentiate with respect to θ $\frac{1}{r}\frac{dr}{d\theta} = -\frac{2}{cosec\frac{\theta}{2}}cosec\frac{\theta}{2}\cot\frac{\theta}{2}$ $\cot\phi_2 = -2\cot\frac{\theta}{2}$

Since $\cot\phi_1 . \cot\phi_2 = -1$, both intersect orthogonally.

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6. **Find the angle of intersection of the following pair of curves:**

(i) $\boldsymbol{r = \sin\theta + \cos\theta}$ **and** $\boldsymbol{r = 2\sin\theta}$

$r = \sin\theta + \cos\theta$ Take log on both sides $\log r = \log\ (\sin\theta + \cos\theta)$ Differentiate w. r. to θ $\frac{1}{r}\frac{dr}{d\theta} = \frac{\cos\theta - \sin\theta}{\sin\theta + \cos\theta}$ $\text{Cot}\,\phi_1 = \tan\left(\frac{\pi}{4} - \theta\right) = \cot\left(\frac{\pi}{4} + \theta\right)$ $\phi_1 = \frac{\pi}{4} + \theta$	$r = 2\sin\theta$ Take log on both sides $\log r = \log(2\sin\theta)$ Differentiate w. r. to θ $\frac{1}{r}\frac{dr}{d\theta} = \frac{2\cos\theta}{2\sin\theta}$ $\cot\phi_2 = \cot\theta$ $\phi_2 = \theta$

Therefore, $|\phi_1 - \phi_2| = \frac{\pi}{4}$

(ii) $\boldsymbol{r = a(1 - \cos\theta)}$ **and** $\boldsymbol{r = 2a\cos\theta}$

$r = a(1 - \cos\theta)$ Take log on both sides $\log r = \log a + \log\ (1 - \cos\theta)$ Differentiate w. r. to θ $\frac{1}{r}\frac{dr}{d\theta} = \frac{\sin\theta}{1 - \cos\theta}$ $\cot\phi_1 = \cot\frac{\theta}{2}$ $\phi_1 = \frac{\theta}{2}$	$r = 2a\cos\theta$ Take log on both sides $\log r = \log 2a + \log\cos\theta$ Differentiate w. r. to θ $\frac{1}{r}\frac{dr}{d\theta} = -\frac{\sin\theta}{\cos\theta} = -\tan\theta$ $\cot\phi_2 = \cot\left(\frac{\pi}{2} + \theta\right)$ $\phi_2 = \frac{\pi}{2} + \theta$

But By data, $1 - \cos\theta = 2\,cos\theta \Rightarrow \theta = \cos^{-1}\frac{1}{3}$

Therefore, $|\phi_1 - \phi_2| = \frac{\pi}{2} + \frac{\theta}{2} = \boldsymbol{\frac{\pi}{2} + \frac{1}{2}}\cos^{-1}\left(\frac{1}{3}\right)$

(iii) $\boldsymbol{r = a\log\theta}$ **and** $\boldsymbol{r = \frac{a}{\log\theta}}$ **(May 22)**

$r = a\log\theta$ Take log on both sides $\log r = \log a + \log\ (\log\theta)$ Differentiate w. r. to θ $\frac{1}{r}\frac{dr}{d\theta} = \frac{1}{\theta\log\theta}$ $\cot\phi_1 = \frac{1}{\theta\log\theta}$ $\tan\phi_1 = \theta\log\theta$	$r = \frac{a}{\log\theta}$ Take log on both sides $\log r = \log a - \log\ (\log\theta)$ Differentiate w. r. to θ $\frac{1}{r}\frac{dr}{d\theta} = -\frac{1}{\theta\log\theta}$ $\cot\phi_2 = -\frac{1}{\theta\log\theta}$ $\tan\phi_2 = -\theta\log\theta$

But by data, $a\log\theta = \frac{a}{\log\theta} \Rightarrow \theta = e$.

Therefore, $\tan(\phi_1 - \phi_2) = \frac{\tan\phi_1 - tan\phi_2}{1+\tan\phi_1.\tan\phi_2} = \frac{e+e}{1+e^2} = \frac{2e}{1+e^2}$

Therefore, $|\phi_1 - \phi_2| = \tan^{-1}\frac{2e}{1+e^2} = 2\tan^{-1} e$

(iv) $\boldsymbol{r = a\sin 2\theta}$ **and** $\boldsymbol{r = a\cos 2\theta}$

$r = a\sin 2\theta$ Take log on both sides $\log r = \log a + \log\ (\sin 2\theta)$ Differentiate w. r. to θ $\frac{1}{r}\frac{dr}{d\theta} = 2\cot 2\theta$ $\cot\phi_1 = 2\cot 2\theta$ $\tan\phi_1 = \frac{1}{2}\tan 2\theta$	$r = a\cos 2\theta$ Take log on both sides $\log r = \log a + \log\ (\cos 2\theta)$ Differentiate w. r. to θ $\frac{1}{r}\frac{dr}{d\theta} = -2\tan 2\theta$ $\cot\phi_2 = -2\tan 2\theta$ $\tan\phi_2 = -\frac{1}{2}\cot 2\theta$

But by data, $a\sin 2\theta = a\cos 2\theta \Rightarrow 2\theta = \pi/4$

Therefore, $\tan|\phi_1 - \phi_2| = \left|\frac{\tan\phi_1 - tan\phi_2}{1+\tan\phi_1 . \tan\phi_2}\right| = \left|\frac{\frac{1}{2}+\frac{1}{2}}{1-\frac{1}{2}.\frac{1}{2}}\right| = \frac{4}{3}$

Therefore, $|\phi_1 - \phi_2| = \tan^{-1}\frac{4}{3}$

(v) $\boldsymbol{r = \frac{a\theta}{1+\theta}}$ **and** $\boldsymbol{r = \frac{a}{1+\theta^2}}$

$r = \frac{a\theta}{1+\theta}$ Take log on both sides $\log r = \log a\theta - \log\ (1+\theta)$ Differentiate w. r. to θ $\frac{1}{r}\frac{dr}{d\theta} = \frac{1}{a\theta}(a) - \frac{1}{1+\theta}$ $\cot\phi_1 = \frac{1}{\theta} - \frac{1}{1+\theta} = \frac{1}{\theta(1+\theta)}$ $\tan\phi_1 = \theta(1+\theta)$	$r = \frac{a}{1+\theta^2}$ Take log on both sides $\log r = \log a - \log\ (1+\theta^2)$ Differentiate w. r. to θ $\frac{1}{r}\frac{dr}{d\theta} = -\frac{2\theta}{1+\theta^2}$ $\cot\phi_2 = -\frac{2\theta}{1+\theta^2}$ $\tan\phi_2 = -\frac{1+\theta^2}{2\theta}$

By data, $\frac{a\theta}{1+\theta} = \frac{a}{1+\theta^2}$, $\theta + \theta^3 = 1 + \theta$, Therefore, $\theta = 1$.

$\tan\phi_1 = 2, \tan\phi_2 = -1, \quad \tan(\phi_1 - \phi_2) = \frac{2+1}{1+2} = 1$

$\phi_1 - \phi_2 = \frac{\pi}{4}$

(vi) $r^2 \sin 2\theta = 4$ and $r^2 = 16 \sin 2\theta$

$r^2 \sin 2\theta = 4$ Take log on both sides $\log r^2 + \log \sin 2\theta = \log 4$ Differentiate w. r. to θ $\frac{2}{r}\frac{dr}{d\theta} + \frac{2\cos 2\theta}{\sin 2\theta} = 0$ $\cot \phi_1 = -\cot 2\theta$ $\cot \phi_1 = \cot(-2\theta)$ $\phi_1 = -2\theta$	$r^2 = 16 \sin 2\theta$ Take log on both sides $\log r^2 = \log 16 + \log \sin 2\theta$ Differentiate w. r. to θ $\frac{2}{r}\frac{dr}{d\theta} = 0 + \frac{2\cos 2\theta}{\sin 2\theta}$ $\cot \phi_2 = \cot 2\theta$ $\phi_2 = 2\theta$

By data, $16 \sin^2 2\theta = 4$, $\sin 2\theta = \frac{1}{2}$, $2\theta = \frac{\pi}{6}, \theta = \frac{\pi}{12}$

Therefore, $|\phi_1 - \phi_2| = 4\theta = 4\left(\frac{\pi}{12}\right) = \frac{\pi}{3}$

1.2 Pedal equations

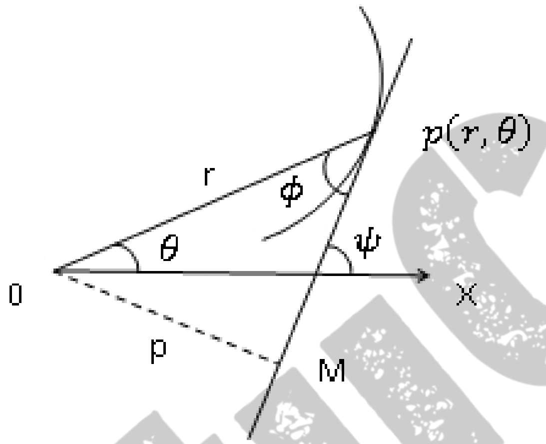
Introduction:

If p is the perpendicular distance from the pole to the tangent of the polar curve, then the equation of the curve in terms of p and r is called pedal equation or $p-r$ equation.

$p-r$ equation is $p = r\, sin\phi$ or $\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2$.

Problems:

1. **With usual notations, prove that $\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2$ and hence deduce that $\frac{1}{p^2} = u^2 + u^4\left(\frac{dr}{d\theta}\right)^2$, where $u = \frac{1}{r}$.**



Let $P(r,\theta) -$ Any point on the polar curve $r = f(\theta)$.

Let r and p – Radius vector and perpendicular distance from the origin respectively.

By diagram, $\frac{p}{r} = \sin\phi \Rightarrow p = r\sin\phi$.

$$\frac{1}{p^2} = \frac{1}{r^2}\, cosec^2\phi = \frac{1}{r^2}(1+\cot^2\phi) = \frac{1}{r^2}\left(1 + \frac{1}{r^2}\left(\frac{dr}{d\theta}\right)^2\right) = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2$$

Therefore, $\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2$

If $u = \frac{1}{r}$, $\frac{du}{d\theta} = -\frac{1}{r^2}\left(\frac{dr}{d\theta}\right)$.

$$\frac{1}{p^2} = u^2 + \left(\frac{du}{d\theta}\right)^2$$

2. **Find the pedal equation of the following curves:**

(i) $\boldsymbol{r^2 = a^2 \sin^2 \theta}$

To find: ϕ	**To find:** Pedal equation
$r^2 = a^2 \sin^2 \theta$	$p = r \sin \phi$
Take log on both sides,	$p = r \sin \theta$
$2 \log r = 2 \log a + 2 \log \sin \theta$	$p^2 = r^2 \sin^2 \theta$
$\log r = \log a + \log \sin \theta$	$p^2 = r^2 \left(\frac{r^2}{a^2}\right)$
$\frac{1}{r}\frac{dr}{d\theta} = \frac{\cos \theta}{\sin \theta}$	$a^2 p^2 = r^4$
$\cot \phi = \cot \theta$	$ap = r^2$
$\phi = \theta$	

(ii) $\boldsymbol{r = 2(1 + \cos \theta)}$

To find: ϕ	**To find:** Pedal equation
$r = 2(1 + \cos \theta)$	$p = r \sin \phi$
Take log on both sides,	$p = r \sin\left(\frac{\pi}{2} + \frac{\theta}{2}\right)$
$\log r = \log 2 + \log(1 + \cos \theta)$	$p^2 = r^2 \cos^2 \frac{\theta}{2}$
$\log r = \log 2 + \log(1 + \cos \theta)$	$p^2 = r^2 \left(\frac{r}{4}\right)$
$\frac{1}{r}\frac{dr}{d\theta} = 0 + \frac{-\sin \theta}{1+\cos \theta} = \frac{-2 \sin\frac{\theta}{2} \cos\frac{\theta}{2}}{2 \cos^2\frac{\theta}{2}}$	$4p^2 = r^3$
$\cot \phi = -\tan \frac{\theta}{2} = \cot\left(\frac{\pi}{2} + \frac{\theta}{2}\right)$	
$\phi = \frac{\pi}{2} + \frac{\theta}{2}$	

(iii) $r^n = a^n \cos n\theta$ (May 22)

To find: ϕ	**To find:** Pedal equation
$r^n = a^n \cos n\theta$ Take log on both sides, $n \log r = n \log a + \log \cos n\theta$ Differentiate w. r. to θ $\frac{n}{r}\frac{dr}{d\theta} = 0 + \frac{-n \sin n\theta}{\cos n\theta}$ $\frac{1}{r}\frac{dr}{d\theta} = -\tan n\theta$ $\cot\phi = \cot\left(\frac{\pi}{2} + n\theta\right)$ $\phi = \frac{\pi}{2} + n\theta$	$p = r \sin\phi$ $p = r\sin\left(\frac{\pi}{2} + n\theta\right)$ $p = r\cos n\theta$ $p = r\left(\frac{r^n}{a^n}\right)$ $a^n p = r^{n+1}$

(iv) $r^m \cos m\theta = a^m$

To find: ϕ	**To find:** Pedal equation
$r^m \cos m\theta = a^m$ Take log on both sides, $m \log r + \log \cos m\theta = m \log a$ Differentiate w. r. to θ $\frac{m}{r}\frac{dr}{d\theta} + \frac{-m \sin m\theta}{\cos m\theta} = 0$ $\frac{1}{r}\frac{dr}{d\theta} = \tan m\theta$ $\cot\phi = \cot\left(\frac{\pi}{2} - m\theta\right)$	$p = r \sin\phi$ $p = r\sin\left(\frac{\pi}{2} - m\theta\right)$ $p = r\cos m\theta$ $p = r\left(\frac{a^m}{r^m}\right)$ $r^{m-1} p = a^m$

$\phi = \frac{\pi}{2} - m\theta$	

(v) $r^m = a^m(\cos m\theta + \sin m\theta)$

To find: ϕ	**To find:** Pedal equation
$r^m = a^m(\cos m\theta + \sin m\theta)$ Take log on both sides, $m\log r = \log a^m + \log(\cos m\theta + \sin m\theta)$ Differentiate w. r. to θ $\frac{m}{r}\frac{dr}{d\theta} = 0 + \frac{-m\sin m\theta + m\cos m\theta}{\cos m\theta + \sin m\theta}$ $\frac{1}{r}\frac{dr}{d\theta} = \tan\left(\frac{\pi}{4} - m\theta\right)$ $\cot\phi = \cot\left(\frac{\pi}{2} - \frac{\pi}{4} + m\theta\right)$ $\phi = \frac{\pi}{4} + m\theta$	$p = r\sin\phi$ $p = r\sin\left(\frac{\pi}{4} + m\theta\right)$ $p = \frac{r}{\sqrt{2}}(\cos m\theta + \sin m\theta)$ $p = \frac{r}{\sqrt{2}}\left(\frac{r^m}{a^m}\right)$ $\sqrt{2}a^m p = r^{m+1}$

3. **Find the Pedal equation of the following curves:**

(i) $r = ae^{m\theta}$

To find: ϕ	**To find:** Pedal equation
$r = ae^{m\theta}$ Take log on both sides, $\log r = \log a + m\theta$ Differentiate w. r. to θ $\frac{1}{r}\frac{dr}{d\theta} = 0 + m$ $\cot\phi = m$	$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2$ $\frac{1}{p^2} = \frac{1}{r^2}(1 + m^2)$ $r^2 = p^2(1 + m^2)$

(ii) $\frac{l}{r} = 1 + e\cos\theta$

To find: ϕ	**To find:** Pedal equation
$\frac{l}{r} = 1 + e\cos\theta$ Take log on both sides, $\log l - \log r = \log(1 + e\cos\theta)$ Differentiate w. r. to θ $0 - \frac{1}{r}\frac{dr}{d\theta} = \frac{-e\sin\theta}{1+e\cos\theta}$ $\cot\phi = \frac{e\sin\theta}{1+e\cos\theta}$	$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2$ $\frac{1}{p^2} = \frac{1}{r^2}\left(1 + \frac{e^2\sin^2\theta}{(1+e\cos\theta)^2}\right)$ $r^2(1 + e\cos\theta)^2 = p^2(1 + e^2 + 2e\cos\theta)$ $l^2 = p^2\left(1 + e^2 + 2\left(\frac{l}{r} - 1\right)\right)$ $l^2 = p^2\left(e^2 + \frac{2l}{r} - 1\right)$

(iii) $\frac{2a}{r} = 1 - \cos\theta$

To find: ϕ	**To find:** Pedal equation
$\frac{2a}{r} = 1 - \cos\theta$ Take log on both sides, $\log 2a - \log r = \log(1 - \cos\theta)$ Differentiate w. r. to θ $0 - \frac{1}{r}\frac{dr}{d\theta} = \frac{\sin\theta}{1-\cos\theta} = \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\sin^2\frac{\theta}{2}}$ $\cot\phi = -\cot\frac{\theta}{2} = \cot\left(\pi - \frac{\theta}{2}\right)$ $\phi = \pi - \frac{\theta}{2}$	$p = r\sin\phi$ $p = r\sin\left(\pi - \frac{\theta}{2}\right) = r\sin\frac{\theta}{2}$ $\frac{p^2}{r^2} = \sin^2\frac{\theta}{2} = \frac{a}{r}$ $p^2 = ar$

1.3 Radius of curvature

Introduction:

The reciprocal of the curvature of a curve at any point p is called the radius of curvature at p.

It is defined by $\rho = \frac{ds}{d\psi}$.

Cartesian form	Polar form
$\rho = \frac{(1+y_1^2)^{3/2}}{y_2}$	$\rho = \frac{(r^2+r_1^2)^{3/2}}{r^2+2r_1^2-r\,r_2}$
Parametric form	**Pedal form**
$\rho = \frac{(x'^2+y'^2)^{\frac{3}{2}}}{x'y''-x''y'}$	$\rho = r\frac{dr}{dp}$

1. **Derive radius of curvature for the Cartesian curve $y = f(x)$.** **(May 22)**

$tan\psi = \frac{dy}{dx}$

$tan\psi = y_1$

$\psi = tan^{-1}(y_1)$

Differentiating w. r. to x,

$\frac{d\psi}{dx} = \frac{1}{1+y_1^2} . y_2$

Therefore, radius of curvature is given by

$\rho = \frac{ds}{d\psi} = \frac{ds}{dx} . \frac{dx}{d\psi} = \sqrt{1+y_1^2} . \frac{1+y_1^2}{y_2}$

$\rho = \frac{(1+y_1^2)^{\frac{3}{2}}}{y_2}$

$$\boxed{\rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}}$$

2. Derive radius of curvature for the parametric curve $x = f(t), y = g(t)$.

$y_1 = \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{y'}{x'}$	$y_2 = \frac{d}{dx}\left(\frac{y'}{x'}\right) = \frac{d}{dt}\left(\frac{y'}{x'}\right)\left(\frac{dt}{dx}\right) = \left(\frac{x'y''-x''y'}{x'^2}\right)\left(\frac{1}{x'}\right)$

Therefore, radius of curvature is given by

$$\rho = \frac{(1+y_1^2)^{\frac{3}{2}}}{y_2} = \frac{\left(1+\frac{y'^2}{x'^2}\right)^{\frac{3}{2}}}{\left(\frac{x'y''-x''y'}{x'^3}\right)}$$

$$\boxed{\rho = \frac{(x'^2+y'^2)^{\frac{3}{2}}}{x'y''-x''y'}}$$

3. Derive radius of curvature for the polar curve $r = f(\theta)$.

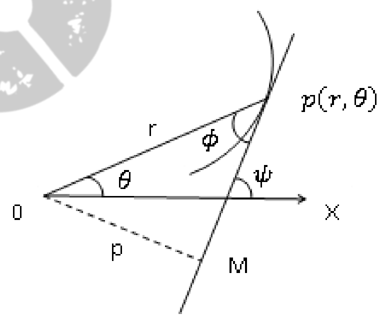
By diagram, $\chi = \theta + \phi$

$$\frac{d\chi}{ds} = \frac{d\theta}{ds} + \frac{d\phi}{ds}$$

$$= \frac{d\theta}{ds} + \frac{d\phi}{d\theta}\cdot\frac{d\theta}{ds}$$

$$= \frac{d\theta}{ds}\left(1 + \frac{d\phi}{d\theta}\right)$$

$$= \frac{1+\frac{d\phi}{d\theta}}{\frac{ds}{d\theta}} \text{ ---------- (1)}$$



But $\tan\phi = r\frac{d\theta}{dr}$

$$\phi = \tan^{-1}\left(\frac{r}{r_1}\right)$$

$$\frac{d\phi}{d\theta} = \frac{1}{1+\left(\frac{r}{r_1}\right)^2}\cdot\frac{r_1.r_1 - rr_2}{r_1^2} = \frac{r_1^2-rr_2}{r^2+r_1^2}$$

$$1 + \frac{d\phi}{d\theta} = 1 + \frac{r_1^2-rr_2}{r^2+r_1^2} = \frac{r^2+2r_1^2-rr_2}{r^2+r_1^2}$$

$$\frac{ds}{d\theta} = \sqrt{r^2+r_1^2}$$

Therefore, radius of curvature is given by

$$\frac{1}{\rho} = \frac{d\chi}{ds}$$

$$= \frac{1+\frac{d\phi}{d\theta}}{\frac{ds}{d\theta}} \quad \text{[By (1)]}$$

$$= \frac{1}{\sqrt{r^2+r_1^2}}\cdot\frac{r^2+2r_1^2-rr_2}{r^2+r_1^2}$$

$$\boxed{\rho = \frac{(r^2+r_1^2)^{\frac{3}{2}}}{r^2+2r_1^2-rr_2}}$$

Note:

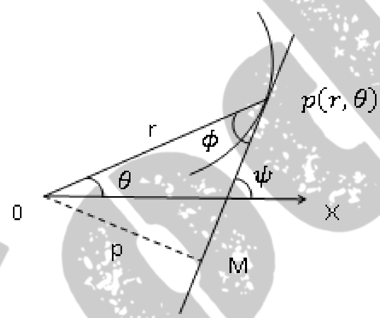
$\frac{ds}{dr} = \frac{ds}{d\theta} \cdot \frac{d\theta}{dr} = \left(\sqrt{r^2 + r_1^2}\right) \frac{1}{r_1} = \sqrt{r^2 \left(\frac{d\theta}{dr}\right)^2 + 1}$

$\frac{ds}{dr} = \sqrt{1 + \tan^2 \phi} = \sec \phi$

$\cos \phi = \frac{dr}{ds}$

$\sin \phi = \tan \phi \, . \cos \phi = r \frac{d\theta}{dr} \, . \frac{dr}{ds} = r \frac{d\theta}{ds}$

4. **Derive radius of curvature for the pedal curve $p = f(r)$.**



By diagram, $\chi = \theta + \phi$

$\frac{d\chi}{ds} = \frac{d\theta}{ds} + \frac{d\phi}{ds}$

But $p = r \sin \phi$

$\frac{dp}{dr} = \sin \phi + r \cos \phi \frac{d\phi}{dr}$

$= r \frac{d\theta}{ds} + r \frac{dr}{ds} \, . \frac{d\phi}{dr}$

$= r \left(\frac{d\theta}{ds} + \frac{d\phi}{ds}\right)$

$= r \left(\frac{d\chi}{ds}\right)$

$\boxed{\rho = r \frac{dr}{dp}}$

5. **Find the radius of curvature for $x^4 + y^4 = 2$ at $(1, 1)$ (July '16)**

$x^4 + y^4 = 2$

Differentiate w. r. to x,

$4x^3 + 4y^3 y' = 0$

$x^3 + y^3 y' = 0$ -------- (1)

Differentiate again w. r. to x,

$3x^2 + 3y^2 (y')^2 + y^3 y" = 0$ -------- (2)

At $(1, 1), y' = -1, y'' = -6.$

Radius of curvature is given by

$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2} = \frac{(1+1)^{\frac{3}{2}}}{-6} = \frac{2\sqrt{2}}{-6} = \frac{\sqrt{2}}{-3}$$

6. **Find the radius of curvature of the Folium $x^3 + y^3 = 3axy$ at the point $(3a/2,\ 3a/2)$.**

$x^3 + y^3 = 3axy$

Differentiate with respect to x

$x^2 + y^2y' = a(xy' + y)$ ------ (1)

Differentiate again with respect to x

$2x + 2y(y')^2 + y^2y" = a\,(xy" + 2y')$ ------- (2)

At $\left(\frac{3a}{2}, \frac{3a}{2}\right)$,

$(1) \Rightarrow \frac{9a^2}{4} + \frac{9a^2}{4}y' = a\left(\frac{3a}{2}y' + \frac{3a}{2}\right)$

$\left(\frac{9a^2}{4} - \frac{3a^2}{2}\right)y' = \frac{3a^2}{2} - \frac{9a^2}{4}$, $\boldsymbol{y' = -1}$

$(2) \Rightarrow 2\left(\frac{3a}{2}\right) + 2\left(\frac{3a}{2}\right) + \left(\frac{3a}{2}\right)^2 y" = a\left(\frac{3a}{2}y" - 2\right)$

$$3a + 3a + \frac{9a^2}{4}y'' - \frac{3a^2}{2}y'' = -2a$$

$\frac{3a^2}{4}y'' = -8a$, $\boldsymbol{y" = \frac{-32}{3a}}$

Radius of curvature is given by

$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2} = \left(\frac{2^{\frac{3}{2}}}{-32}\right)3a = -\frac{3a\sqrt{2}}{16}$$

7. **Find the radius of curvature of the catenary $y = c\,cosh\frac{x}{c}$ at $(c, 0)$.**

$y = c\,cosh\frac{x}{c}$

Differentiate with respect to x,

$y_1 = \sinh\frac{x}{c}$ --------- (1)

Differentiate again with respect to x,

$y_2 = \frac{1}{c}\cosh\frac{x}{c}$ --------- (2)

$At\ (c, 0)$, $y_1 = \sinh 1$, $y_2 = \frac{1}{c}\cosh 1$

Radius of curvature is given by

$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2} = \frac{(1+\sinh^2 1)^{3/2}}{\frac{1}{c}\cosh 1} = c\cosh^2 1 = \frac{y^2}{c}$$

8. Find the radius of curvature of the parabola $y^2 = 4ax$ at $(at^2, 2at)$.

$y^2 = 4ax$

Differentiate w. r. to x,

$yy_1 = 2a$ ---------- (1)

Differentiate again w. r. to x,

$y_1^2 + yy_2 = 0$ ---------- (2)

At $(at^2, 2at)$,

$y_1 = \frac{1}{t}$, $y_2 = -\frac{1}{2at^3}$

Radius of curvature is given by

$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2} = \frac{\left(1+\frac{1}{t^2}\right)^{3/2}}{\left(-\frac{1}{2at^3}\right)} = -2a(1+t^2)^{\frac{3}{2}}$$

9. Find the radius of curvature of the curve $y = x^3(x-a)$ at $(a, 0)$.

$y = x^3(x-a)$

Differentiate w. r. to x,

$y_1 = x^3 - (x-a)3x^2$---------- (1)

Differentiate again w. r. to x,

$y_2 = 6x^2 + 6x(x-a)$ ---------- (2)

$at\ (a, 0)$, $y_1 = a^3$, $y_2 = 6a^2$

Radius of curvature is given by

$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2} = \frac{(1+a^6)^{\frac{3}{2}}}{6a^2}$$

10. For the cardioid $r = a(1-\cos\theta)$, show that $\frac{\rho^2}{r}$ is a constant.

Step 1: Find ϕ	**Step 3: Find radius of curvature**
$r = a(1-\cos\theta)$	$r^3 = 2ap^2$
Take log on both sides,	Differentiate w.r.to p
$\log r = \log a + \log(1-\cos\theta)$	$3r^2\frac{dr}{dp} = 4ap$
Differentiate w. r. to x,	$r\frac{dr}{dp} = \frac{4ap}{3r}$
$\frac{1}{r}\frac{dr}{d\theta} = \frac{\sin\theta}{1-\cos\theta}$	$\rho = \frac{4ap}{3r}$
$\cot\phi = \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\sin^2\frac{\theta}{2}}$	$\rho^2 = \frac{16a^2p^2}{9r^2} = \frac{8ar}{9}$
$\cot\phi = \cot\frac{\theta}{2}$	Therefore,
$\phi = \frac{\theta}{2}$	$\frac{\rho^2}{r}$ is a constant.
Step 2: Find $p-r$ equation	
$p = r\sin\phi = r\sin\frac{\theta}{2}$	
$p^2 = r^2\sin^2\frac{\theta}{2}$	
$p^2 = \frac{r^3}{2a}$	
$r^3 = 2ap^2$	

11. Find the radius of curvature for $y^2 = \frac{a^2(a-x)}{x}$ where the curve meets the x-axis.

$y^2 = \frac{a^2(a-x)}{x}$

$xy^2 = a^3 - a^2x$ ---- (1)

Differentiate w.r.to x ,

$2xyy' + y^2 = -a^2$

$2xyy' = -\frac{a^3}{x}$

$\frac{dy}{dx} = -\frac{a^3}{2yx^2}$ does not exist at $y = 0$.

$\therefore x_1 = \frac{dx}{dy} = -\frac{2x^2y}{a^3}$

$x_2 = \frac{d^2x}{dy^2} = -\frac{2}{a^3}\left(x^2 + 2xy\frac{dx}{dy}\right).$

At $y = 0$,

$x_1 = 0\,, x_2 = -\frac{2x^2}{a^3}$

Therefore, radius of curvature is given by

$\rho = \frac{(1+x_1^2)^{3/2}}{x_2} = \frac{-a^3}{2x^2}$.

12. If $\rho_{1,}$ ρ_2 be the radii of curvature at the extremities of any chord of the cardioid $r = a(1 + cos\theta)$ which passes through the pole, show that ${\rho_1}^2 + {\rho_2}^2 = \frac{16a^2}{9}$

(May 22)

Step 1: Find ϕ $r = a(1 + cos\theta)$ $\log r = \log a + \log(1 + \cos\theta)$ $\frac{1}{r}\frac{dr}{d\theta} = -\frac{\sin\theta}{1+\cos\theta}$ $\cot\phi = -\frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}}$ $= -\tan\frac{\theta}{2}$ $= \cot\left(\frac{\pi}{2} + \frac{\theta}{2}\right)$ $\phi = \frac{\pi}{2} + \frac{\theta}{2}$	**Step 2:** Find $p - r$ equation $p = r\sin\phi$ $= r\sin\left(\frac{\pi}{2} + \frac{\theta}{2}\right)$ $p^2 = r^2\cos^2\frac{\theta}{2}$ $p^2 = \frac{r^3}{2a}\ \left[\because By\ data, r = 2a\cos^2\frac{\theta}{2}\right]$ $r^3 = 2ap^2$

Step 3: Find ρ $r^3 = 2ap^2$ Differentiate w.r.to p $3r^2\frac{dr}{dp} = 4ap$ $r\frac{dr}{dp} = \frac{4ap}{3r}$ $\rho = \frac{4ap}{3r}$ $\rho^2 = \frac{16a^2p^2}{9r^2}$ $= \frac{8ar}{9}$	**Step 4:** To prove $\rho_1^2 + \rho_2^2 = \frac{16a^2}{9}$ At (r, θ), $\rho_1^2 = \frac{8ar}{9} = \frac{8a^2}{9}(1 + \cos\theta)$ At $(r, \pi + \theta)$, $\rho_2^2 = \frac{8ar}{9} = \frac{8a^2}{9}(1 - \cos\theta)$. Adding both. $\rho_1^2 + \rho_2^2 = \frac{16a^2}{9}$

13. Show that the radius of curvature at any point of the cycloid

$$\boldsymbol{x = a(\theta + \sin\theta),\ \ y = a(1 - \cos\theta)\ \textit{is}\ 4a\cos\left(\frac{\theta}{2}\right)}$$

$\frac{dx}{d\theta} = a(1 + \cos\theta),\ \frac{dy}{d\theta} = a\sin\theta$

$y_1 = \frac{dy}{dx}$ $= \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$ $= \frac{a\sin\theta}{a(1+\cos\theta)}$ $= \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}} = \tan\frac{\theta}{2}$	$y_2 = \frac{d^2y}{dx^2}$ $= \frac{1}{2}\sec^2\frac{\theta}{2} \times \frac{d\theta}{dx}$ $= \frac{\frac{1}{2}\sec^2\frac{\theta}{2}}{a(1+\cos\theta)}$ $= \frac{\frac{1}{2}\sec^2\frac{\theta}{2}}{a\left(2\cos^2\frac{\theta}{2}\right)} = \frac{1}{4a}\sec^4\frac{\theta}{2}$

Therefore, radius of curvature is given by

$$\rho = \frac{(1+y_1^2)^{\frac{3}{2}}}{y_2} = \frac{\sec^3\frac{\theta}{2}}{\frac{1}{4a}\sec^4\frac{\theta}{2}} = 4a\cos\frac{\theta}{2}$$

14. Show that the radius of curvature at any point of the cycloid

$$\boldsymbol{x = a(\theta - \sin\theta),\ \ y = a(1 - \cos\theta)\ \textit{is}\ 4a\sin\left(\frac{\theta}{2}\right)}$$

$\frac{dx}{d\theta} = a(1 - \cos\theta),\ \frac{dy}{d\theta} = a\sin\theta$

$y_1 = \frac{dy}{dx}$ $= \frac{a\sin\theta}{a(1-\cos\theta)}$ $= \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\sin^2\frac{\theta}{2}}$ $= \cot\frac{\theta}{2}$	$y_2 = \frac{d^2y}{dx^2}$ $= -\frac{1}{2}cosec^2\left(\frac{\theta}{2}\right) \times \frac{d\theta}{dx}$ $= -\frac{1}{2}cosec^2\left(\frac{\theta}{2}\right) \times \frac{1}{a(1-\cos\theta)}$ $= -\frac{1}{4a}cosec^4\left(\frac{\theta}{2}\right)$

Therefore, radius of curvature is given by

$$\rho = \frac{(1+y_1^2)^{\frac{3}{2}}}{y_2} = \frac{\left(1+\cot^2\frac{\theta}{2}\right)^{\frac{3}{2}}}{-\frac{1}{4a}cosec^4\left(\frac{\theta}{2}\right)} = -4a\sin\frac{\theta}{2}$$

Ignoring sign, $\rho = 4a\sin\frac{\theta}{2}$

15. Show that the radius of curvature at any point of the cycloid $\boldsymbol{x = a\cos^3 t, y = a\sin^3 t}$ at $\boldsymbol{t = \frac{\pi}{4}}$.

$\frac{dx}{d\theta} = -3a\cos^2 t\sin t\,,\ \frac{dy}{d\theta} = 3a\sin^2 t\cos t$

$y_1 = \frac{dy}{dx}$ $= \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$ $= \frac{3a\sin^2 t\cos t}{-3a\cos^2 t\sin t}$ $= -\tan t$	$y_2 = \frac{d^2y}{dx^2}$ $= -\sec^2 t \times \frac{dt}{dx}$ $= \frac{\sec^2 t}{3a\cos^2 t\sin t}$ $= \frac{1}{3a}\frac{\sec^4 t}{\sin t}$

At $t = \frac{\pi}{4}$, $y_1 = -1, y_2 = \frac{1}{3a}4\sqrt{2}$

Therefore, radius of curvature is given by

$$\rho = \frac{(1+y_1^2)^{\frac{3}{2}}}{y_2} = \frac{(1+1)^{\frac{3}{2}}}{\frac{1}{3a}4\sqrt{2}} = \frac{3a}{2}$$

16. Find the radius of curvature of the curve $\boldsymbol{x = a(\cos t + t\sin t), y = a(\sin t - t\cos t)}$ at any point $\boldsymbol{t}$.

$\frac{dx}{dt} = a(-\sin t + \sin t + t\cos t) = at\cos t,\quad \frac{dy}{dt} = a(\cos t + t\sin t - \cos t) = at\sin t$

$y_1 = \frac{dy}{dx}$ $= \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$ $= \frac{at\sin t}{at\cos t}$ $= \tan t$	$y_2 = \frac{d^2y}{dx^2}$ $= \sec^2 t \times \frac{dt}{dx}$ $= \frac{\sec^2 t}{at\cos t}$ $= \frac{1}{at}\sec^3 t$

Therefore, radius of curvature is given by

$$\rho = \frac{(1+y_1^2)^{\frac{3}{2}}}{y_2} = \frac{(1+\tan^2 t)^{\frac{3}{2}}}{\frac{1}{3a}4\sqrt{2}} = \frac{\sec^3 t}{\frac{1}{at}\sec^3 t} = at.$$