Module 2 - Series expansion and multivariable calculus

2.1 Taylor's and Maclaurin's series for one variable

Introduction:

Taylor's series: $f(x) = f(a) + \frac{x-a}{1!}f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \cdots$

Maclaurin's series: $f(x) = f(0) + \frac{x}{1!}f'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \cdots$

Maclaurin's series is a Taylor's series expansion of a function at the origin.

***Problems**:*

1. **Obtain the Maclaurin's series expansion for the following functions:**

$$\boldsymbol{\sqrt{1+\sin 2x} = 1 + x - \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} + \cdots}$$ **(May 22)**

$$f(x) = \sqrt{1+\sin 2x} = \sqrt{1+2\sin x\cos x} = \sqrt{\sin^2 x + \cos^2 x + 2\sin x\cos x} = \sqrt{(\sin x + \cos x)^2}$$

$y = \sin x + \cos x$	$f(0) = 1$
$y_1 = \cos x - \sin x$	$f'(0) = 1$
$y_2 = -\sin x - \cos x = -y$	$f''(0) = -1$
$y_3 = -y_1$	$f'''(0) = -1$
$y_4 = -y_2$	$f^{iv}(0) = 1$
$y_5 = -y_3$	$f^{v}(0) = 1$

By Maclaurin's series,

$$f(x) = f(0) + \frac{x}{1!}f'(0) + \frac{x^2}{2!}f''(0) + \cdots$$

$$\sqrt{1+\sin 2x} = 1 + \frac{x}{1!}(1) + \frac{x^2}{2!}(-1) + \frac{x^3}{3!}(-1) + \frac{x^4}{4!}(1) + \cdots$$

$$\Rightarrow \sqrt{1+\sin 2x} = 1 + x - \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

2. $\sec x$ upto x^4 term

$y = \sec x$ | $f(0) = 1$

$y_1 = \sec x \tan x = y \tan x$ | $f'(0) = 0$

$y_2 = y \sec^2 x + y_1 \tan x = y^3 + y \tan^2 x$ | $f''(0) = 1$

$\quad = y^3 + y(\sec^2 x - 1) = y^3 + y^3 - y$

$\quad = 2y^3 - y$

$y_3 = 6y^2 y_1 - y_1$ | $f'''(0) = 0$

$y_4 = 12yy_1^2 + 6y^2 y_2 - y_2$ | $f^{iv}(0) = 5$

By Maclaurin's series,

$$f(x) = f(0) + \frac{x}{1!}f'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \frac{x^4}{4!}f^{iv}(0) + \ \dots$$

$$\sec x = 1 + 0 + \frac{x^2}{2!}(1) + 0 + \frac{x^4}{4!}(5) + \cdots = 1 + \frac{x^2}{2} + \frac{5x^4}{24}$$

3. $\log(1 + e^x)$ up to 3rd degree term.

$y = \log(1 + e^x)$ | $f(0) = \log 2$

$y_1 = \frac{e^x}{1+e^x}$ | $f'(0) = \frac{1}{2}$

$y_2 = \frac{(1+e^x)e^x - e^x e^x}{(1+e^x)^2} = y_1 - {y_1}^2$ | $f''(0) = \frac{1}{4}$

$y_3 = y_2 - 2y_1 y_2$ | $f'''(0) = 0$

$y_4 = y_3 - 2(y_1 y_3 + y_2^2)$ | $f^{iv}(0) = -\frac{1}{8}$

By Maclaurin's series,

$$f(x) = f(0) + \frac{x}{1!}f'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \frac{x^4}{4!}f^{iv}(0) + \ \dots$$

$$\log(1 + e^x) = \log 2 + \frac{x}{1!}\left(\frac{1}{2}\right) + \frac{x^2}{2!}\left(\frac{1}{4}\right) + \frac{x^3}{3!}(0) + \frac{x^4}{4!}\left(-\frac{1}{8}\right)$$

$$= \log 2 + \frac{x}{2} + \frac{x^2}{8} - \frac{x^4}{192}$$

4. **Prove that** $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$

$y = \log(1+x)$	$f(0) = 0$
$y_1 = \frac{1}{1+x}$	$f'(0) = 1$
$y_2 = -\frac{1}{(1+x)^2} = -y_1^2$	$f''(0) = -1$
$y_3 = -2y_1y_2$	$f'''(0) = 2$
$y_4 = -2y_1y_3 - 2y_2^2$	$f^{iv}(0) = -6$

By Maclaurin's series,

$$f(x) = f(0) + \frac{x}{1!}f'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \frac{x^4}{4!}f^{iv}(0) + \ldots$$

$$\log(1+x) = 0 + \frac{x}{1!} - \frac{x^2}{2!} + \frac{x^3}{3!}(2) + \frac{x^4}{4!}(-6) + \ldots$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

5. $\tan x = x + \frac{2x^3}{3!} + \frac{16x^5}{5!} + \cdots$

$y = \tan x$	$y(0) = 0$
$y_1 = \sec^2 x = 1 + \tan^2 x = 1 + y^2$	$y_1(0) = 1$
$y_2 = 2yy_1 = 2y + 2y^3$	$y_2(0) = 0$
$y_3 = 2y_1^2 + 2yy_2$	$y_3(0) = 2$
$y_4 = 4y_1y_2 + 2y_1y_2 + 2yy_3 = 6y_1y_2 + 2yy_3$	$y_4(0) = 0$
$y_5 = 6y_2^2 + 6y_1y_3 + 2y_1y_3 + 2yy_4$	$y_5(0) = 16$

By Maclaurin's series,

$$f(x) = f(0) + \frac{x}{1!}f'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \cdots$$

$$\tan x = 0 + x + 0 + \frac{x^3}{3} + 0 + \frac{x^5}{5!}(16) + \cdots$$

$$= x + \frac{x^3}{3} + \frac{x^5}{5!}(16) + \cdots$$

6. $\log \sec x = \frac{x^2}{2!} + \frac{2x^4}{4!} + \frac{16x^6}{6!} + \cdots$

$y = \log \sec x$	$y(0) = 0$
$y_1 = \frac{1}{\sec x} \sec x \tan x = \tan x$	$y_1(0) = 0$
$y_2 = \sec^2 x = 1 + \tan^2 x = 1 + y_1^2$	$y_2(0) = 1$
$y_3 = 2y_1y_2$	$y_3(0) = 0$
$y_4 = 2y_2^2 + 2y_1y_3$	$y_4(0) = 2$
$y_5 = 4y_2y_3 + 2y_1y_4 + 2y_2y_3 = 6y_2y_3 + 2y_1y_4$	$y_5(0) = 0$
$y_6 = 6y_2y_4 + 6y_3^2 + 2y_1y_5 + 2y_2y_4$	$y_6(0) = 16$

By Maclaurin's series,

$$f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f'^{v}(0) + \frac{x^5}{5!} f^{v}(0) \ldots$$

$$\log \sec x = 0 + 0 + 0 + \frac{x^2}{2!} + 0 + \frac{x^4}{4!}(2) + 0 + \frac{x^6}{6!}(16) + \cdots$$

$$= \frac{x^2}{2!} + \frac{2x^4}{4!} + \frac{x^6}{6!}(16) + \cdots$$

Home work:

7. $\frac{e^x}{1+e^x}$ **up to 3rd degree term.**

Hint: $y = \frac{e^x}{1+e^x}$, $y_1 = \frac{(1+e^x)e^x - e^{2x}}{(1+e^x)^2} = \frac{e^x}{(1+e^x)^2} = y_1^2$

Ans: $\frac{1}{2} + \frac{x}{4} - \frac{x^3}{48} + \cdots$

8. $e^{\sin x} = 1 + x + \frac{x^2}{2!} - \frac{3x^4}{4!} - \frac{8x^5}{5!} + \cdots$

$Hint: y = 1, y_1 = 1, y_2 = 1, y_3 = 0, y_4 = -3, y_5 = -8$

9. $e^{\tan^{-1} x} = 1 + x + \frac{x^2}{2} - \frac{x^3}{3!} - \frac{7x^4}{4!} + \cdots$

$Hint: y = 1, y_1 = 1, y_2 = 1, y_3 = -1, y_4 = -7, y_5 = 5$

10. $\log(1 + e^x) = \log 2 + \frac{x}{2} + \frac{x^2}{8} - \frac{x^4}{192} + \cdots$

$$Hint: y = \log 2\,, y_1 = \frac{1}{2}, y_2 = \frac{1}{4}, y_3 = 0, y_4 = -\frac{1}{8}$$

2.2 Evaluation of indeterminate forms

Basic results:

- $\lim_{x\to 0}\frac{\sin x}{\mathrm{x}} = \lim_{x\to 0}\frac{\tan x}{\mathrm{x}} = \lim_{x\to 0}\frac{e^{x}-1}{\mathrm{x}} = \lim_{x\to 0}\frac{\log(1+x)}{\mathrm{x}} = 1$
- $\lim_{x\to\infty}\left(1+\frac{1}{\mathrm{x}}\right)^{x} = \lim_{x\to 0}(1+\mathrm{x})^{\frac{1}{\mathrm{x}}} = e$
- $\lim_{x\to 0}\frac{a^{x}-1}{\mathrm{x}} = \log_e a$

Introduction:

- Indeterminate forms: $\frac{0}{0}, \frac{\infty}{\infty}, \infty-\infty, 0\times\infty, 0^0, 1^{\infty}, \infty^0$.
- Limits which lead to indeterminate forms are evaluated by using L' Hospital's rule.
- L' Hospital's rule: Suppose $f(x)$ and $g(x)$ are functions such that

(i) $\lim_{x\to a} f(x) = 0, \quad \lim_{x\to a} g(x) = 0$ (ii) $f'(x) and\ g'(x)$ exist and $g'(a) \neq 0$,

Then $\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$ provided the limit on the RHS exists.

1. Prove that $\lim_{x\to 0}(1+x)^{1/x} = e$

Let $L = \lim_{x\to 0}(1+x)^{1/x}$

$\log L = \lim_{x\to 0}\log(1+x)^{1/x}$

$= \lim_{x\to 0}\frac{1}{x}\log(1+x)$

$= \lim_{x\to 0}\frac{\log(1+x)}{x} = \frac{0}{0}\ form$

By L.H rule,

$\log L = \lim_{x\to 0}\frac{1}{1+x} = 1$

By taking anti log, $L = e$

2. Evaluate $\lim_{x\to 0}\left(\frac{1^x+2^x+3^x}{3}\right)^{\frac{1}{x}}$

Let $L = \lim_{x\to 0}\left(\frac{1^x+2^x+3^x}{3}\right)^{\frac{1}{x}}$

$$\log L = \lim_{x\to 0} \log\left(\frac{1^x+2^x+3^x}{3}\right)^{\frac{1}{x}}$$

$$= \lim_{x\to 0} \frac{1}{x}\log\frac{1^x+2^x+3^x}{3}$$

$$= \lim_{x\to 0} \frac{\log(1^x+2^x+3^x)-\log 3}{x} = \frac{0}{0}\ form$$

By L.H Rule,

$$\log L = \lim_{x\to 0} \frac{1}{1^x+2^x+3^x}\left(1^x\log 1 + 2^x\log 2 + 3^x\log 3\right)$$

$$= \frac{1}{3}(\log 1 + \log 2 + \log 3)$$

$$= \log 6^{\frac{1}{3}}$$

By taking anti log, $L = 6^{\frac{1}{3}}$

3. Prove that $\lim_{x\to 0}\left(\frac{a^x+b^x+c^x}{3}\right)^{\frac{1}{x}} = (abc)^{\frac{1}{3}}$ **(May 22)**

Let $L = \lim_{x\to 0}\left(\frac{a^x+b^x+c^x}{3}\right)^{\frac{1}{x}}$

$$\log L = \lim_{x\to 0} \log\left(\frac{a^x+b^x+c^x}{3}\right)^{\frac{1}{x}}$$

$$= \lim_{x\to 0} \frac{1}{x}\log\frac{a^x+b^x+c^x}{3}$$

$$= \lim_{x\to 0} \frac{\log(a^x+b^x+c^x)-\log 3}{x} = \frac{0}{0}\ form$$

By L.H Rule,

$$\log L = \lim_{x\to 0} \frac{1}{a^x+b^x+c^x}\left(a^x\log a + b^x\log b + c^x\log c\right)$$

$$= \frac{1}{3}(\log a + \log b + \log c)$$

$$= \log(abc)^{\frac{1}{3}}$$

Taking anti log, $L = (abc)^{\frac{1}{3}}$

4. Prove that $\lim\limits_{x\to 0}\left(\frac{tanx}{x}\right)^{\frac{1}{x^2}} = e^{\frac{1}{3}}$

Let $L = \lim\limits_{x\to 0}\left(\frac{tanx}{x}\right)^{\frac{1}{x^2}}$

$$\log L = \lim_{x\to 0}\log\left(\frac{tanx}{x}\right)^{\frac{1}{x^2}}$$

$$= \lim_{x\to 0}\ \frac{1}{x^2}\ \log\frac{tanx}{x} = \frac{0}{0}\ form$$

By L.H rule,

$$\log L = \lim_{x\to 0}\ \frac{1}{2x}\frac{x}{\tan x}\left(\frac{xsec^2x-\tan x}{x^2}\right)$$

$$= \frac{1}{2}\lim_{x\to 0}\ \left(\frac{xsec^2x-\tan x}{x^3}\right) = \frac{0}{0}\ form$$

By L.H rule,

$$\log L = \frac{1}{2}\ \lim_{x\to 0}\ \left(\frac{sec^2x+x.2\sec x.\sec x\tan x-\sec^2 x}{3x^2}\right)$$

$$= \frac{1}{2}\ \lim_{x\to 0}\ \left(\frac{2\sec x.\sec x\tan x}{3x}\right)$$

$$= \frac{1}{3}\ \lim_{x\to 0}\ \left(\frac{\tan x}{x}\right)\sec^2 x = \frac{1}{3}$$

By taking anti log, $L = e^{\frac{1}{3}}$

5. Prove that $\lim_{x \to \frac{\pi}{4}} (\tan x)^{\tan 2x} = \frac{1}{e}$

Let $L = \lim_{x \to \frac{\pi}{4}} (\tan x)^{\tan 2x}$

$$\log L = \lim_{x \to \frac{\pi}{4}} \log(\tan x)^{\tan 2x}$$

$$= \lim_{x \to \frac{\pi}{4}} \tan 2x \ \log \tan x$$

$$= \lim_{x \to \frac{\pi}{4}} \frac{\log \tan x}{\cot 2x} = \frac{0}{0} \ form$$

By L.H rule,

$$\log L = \lim_{x \to \frac{\pi}{4}} \left(\frac{\frac{1}{\tan x} \sec^2 x}{-2 \ cosec^2 \ 2x} \right)$$

$$\log L = -\lim_{x \to \frac{\pi}{4}} \left(\frac{\sin^2 2x}{2 \sin x \cos x} \right)$$

$$= -\lim_{x \to \frac{\pi}{4}} \ \sin 2x = -1$$

Taking anti log, $L = e^{-1} = \frac{1}{e}$

6. Prove that $\lim_{x \to a} \left(2 - \frac{x}{a}\right)^{\tan\left(\frac{\pi x}{2a}\right)} = e^{\frac{2}{\pi}}$

Let $L = \lim_{x \to a} \left(2 - \frac{x}{a}\right)^{tan\left(\frac{\pi x}{2a}\right)}$

$$\log L = \lim_{x \to a} \ \log\left(2 - \frac{x}{a}\right)^{tan\left(\frac{\pi x}{2a}\right)}$$

$$= \lim_{x \to a} tan\left(\frac{\pi x}{2a}\right) \ \log\left(2 - \frac{x}{a}\right)$$

$$= \lim_{x \to a} \frac{\log\left(2 - \frac{x}{a}\right)}{\cot\left(\frac{\pi x}{2a}\right)} = \frac{0}{0} \text{ form}$$

By L.H rule,

$$\log L = \lim_{x \to a} \frac{\left(-\frac{1}{a}\right)\frac{1}{2 - \frac{x}{a}}}{\left(-\frac{\pi}{2a}\right) cosec^2 \frac{\pi x}{2a}}$$

$$\log L = \frac{2}{\pi} \lim_{x \to a} \frac{\sin^2 \frac{\pi x}{2a}}{2 - \frac{x}{a}} = \frac{2}{\pi}$$

Taking anti log, $L = e^{\frac{2}{\pi}}$

7. Prove that $\lim_{x \to \frac{\pi}{2}} (sinx)^{tanx} = 1$

Let $L = \lim_{x \to \frac{\pi}{2}} (\sin x)^{\tan x}$

$$\log L = \lim_{x \to \frac{\pi}{2}} \log(\sin x)^{\tan x}$$

$$= \lim_{x \to \frac{\pi}{2}} \tan x \log \sin x$$

$$= \lim_{x \to \frac{\pi}{2}} \frac{\log \sin x}{\cot x} = \frac{0}{0} \; form$$

By L.H rule,

$$\log L = \lim_{x \to \frac{\pi}{2}} \frac{\cot x}{-cosec^2 x} = 0$$

Taking anti log, $L = e^0 = 1$

8. Prove that $\lim_{x \to \infty} \left(1 + \frac{1}{x^2}\right)^x = 1$

Let $L = \lim_{x \to \infty} \left(1 + \frac{1}{x^2}\right)^x$

$$\log L = \lim_{x \to \infty} \log\left(1 + \frac{1}{x^2}\right)^x$$

$$= \lim_{x \to \infty} x \log\left(1 + \frac{1}{x^2}\right)$$

$$= \lim_{x \to \infty} \frac{\log\left(1 + \frac{1}{x^2}\right)}{\frac{1}{x}} = \frac{0}{0} \; form$$

By L.H rule,

$$\log L = \lim_{x \to \infty} \frac{\frac{1}{1 + \frac{1}{x^2}}\left(-\frac{2}{x^3}\right)}{\left(-\frac{1}{x^2}\right)} = 2 \lim_{x \to \infty} \frac{\frac{1}{x}}{1 + \frac{1}{x^2}} = 0$$

Taking anti log, $L = e^0 = 1.$

Home work:

9. Prove that $\lim_{x\to 0}(a^x + x)^{\frac{1}{x}} = ae$

Hint: $\log L = \lim_{x\to 0}\frac{\log(a^x+x)}{x} = \frac{0}{0}$

$$\log L = \lim_{x\to 0}\frac{a^x \log a+1}{a^x+x} = \log a + 1 = \log a + \log e = \log ae$$

10. Prove that $\lim_{x\to 0}\left(\frac{a^x+b^x+c^x+d^x}{4}\right)^{\frac{1}{x}} = (abcd)^{\frac{1}{4}}$

11. Prove that $\lim_{x\to 0}(\cot x)^{\frac{1}{\log x}} = \frac{1}{e}$

12. Prove that $\lim_{x\to 1}(1-x^2)^{\frac{1}{\log(1-x)}} = e$

13. Prove that $\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^{\frac{1}{x^2}} = e^{-\left(\frac{1}{6}\right)}$

14. Prove that $\lim_{x\to 0}(a^x + x)^{\frac{1}{x}} = ae$

15. Prove that $\lim_{x\to \frac{\pi}{2}}(\cos x)^{\frac{\pi}{2}-x} = 0$

2.4 Partial differentiation

Introduction:

Let $f(x, y)$ be a function of two independent variables x and y.

- **First order partial derivatives:** $f_x = \frac{\partial f}{\partial x},\ f_y = \frac{\partial f}{\partial y}$
- **Second order partial Derivatives:** $f_{xx} = \frac{\partial^2 f}{\partial x^2}$, $f_{yy} = \frac{\partial^2 f}{\partial y^2}$, $f_{xy} = \frac{\partial^2 f}{\partial x \partial y}$, $f_{yx} = \frac{\partial^2 f}{\partial y \partial x}$
- **Property:** $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$

Problems:

1. **If $z = 4x^2 + 8x^3y^2 + 6xy^2 + 8y + 6$. Find $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$.**

   Solution:

   $z = 4x^2 + 8x^3y^2 + 6xy^2 + 8y + 6$ ---- (1)

   Differentiate (1) partially w. r. to x

   $\frac{\partial z}{\partial x} = 8x + 24x^2y^2 + 6y^2$

   Differentiate (1) partially w. r. to y

   $\frac{\partial z}{\partial y} = 16x^3y + 12xy + 8.$

2. **If $z = f(x + ct) + g(x - ct)$, Prove that $\frac{\partial^2 z}{\partial t^2} = c^2 \frac{\partial^2 z}{\partial x^2}$** (MQP)

   Solution:

   $z = f(x + ct) + g(x - ct)$ ---- (1)

   Differentiate (1) partially w. r. to x

   $\frac{\partial z}{\partial x} = f'(x + ct) + g'(x - ct)$

   Differentiate (1) again partially w. r. to x

   $\frac{\partial^2 z}{\partial x^2} = f''(x + ct) + g''(x - ct)$

   Differentiate (1) partially w. r. to t

   $\frac{\partial z}{\partial t} = cf'(x + ct) - cg'(x - ct)$

   Differentiate (1) again partially w. r. to t

   $$\frac{\partial^2 z}{\partial t^2} = c^2 f''(x + ct) + c^2 g''(x - ct)$$
   $$= c^2[f''(x + ct) + g''(x - ct)]$$
   $$= c^2 \frac{\partial^2 z}{\partial x^2}$$

   Therefore,

   $$\frac{\partial^2 z}{\partial t^2} = c^2 \frac{\partial^2 z}{\partial x^2}$$

3. If $z = e^{ax+by}f(ax - by)$ Prove that $b\frac{\partial z}{\partial x} + a\frac{\partial z}{\partial y} = 2abz$ (May 22)

$z = e^{ax+by}f(ax - by)$ ----- (1)

Differentiate (1) partially w. r. to x

$$\frac{\partial z}{\partial x} = ae^{ax+by}f'(ax - by) + ae^{ax+by}f(ax - by)$$

$$= ae^{ax+by}f'(ax - by) + az$$

Differentiate (1) partially w. r. to y

$$\frac{\partial z}{\partial y} = -be^{ax+by}f'(ax - by) + be^{ax+by}f(ax - by)$$

$$= -be^{ax+by}f'(ax - by) + bz$$

Therefore,

$$b\frac{\partial z}{\partial x} + a\frac{\partial z}{\partial y} = abe^{ax+by}f'(ax - by) + abz - abe^{ax+by}f'(ax - by) + abz$$

$$\therefore\ b\frac{\partial z}{\partial x} + a\frac{\partial z}{\partial y} = 2abz$$

Note: $(x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx) = x^3 + y^3 + z^3 - 3xyz$

4. If $u = log(x^3 + y^3 + z^3 - 3xyz)$. then Prove that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3}{x+y+z}$

and hence show that $\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = \frac{-9}{(x+y+z)^2}$

$u = log(x^3 + y^3 + z^3 - 3xyz)$

$$\frac{\partial u}{\partial x} = \frac{3x^2-3yz}{x^3+y^3+z^3-3xyz}$$

$$\frac{\partial u}{\partial y} = \frac{3y^2-3xz}{x^3+y^3+z^3-3xyz}$$

$$\frac{\partial u}{\partial z} = \frac{3z^2-3xy}{x^3+y^3+z^3-3xyz}$$

Case 1:

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3x^2-3yz}{x^3+y^3+z^3-3xyz} + \frac{3y^2-3xz}{x^3+y^3+z^3-3xyz} + \frac{3z^2-3xy}{x^3+y^3+z^3-3xyz}$$

$$= \frac{3(x^2+y^2+z^2-xy-yz-zx)}{x^3+y^3+z^3-3xyz}$$

$$= \frac{3(x^2+y^2+z^2-xy-yz-zx)}{(x+y+z)(x^2+y^2+z^2-xy-yz-zx)}$$

$$= \frac{3}{x+y+z}$$

Case 2:

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)u$$

$$= \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\right)$$

$$= \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{3}{x+y+z}\right) = \frac{-9}{(x+y+z)^2}$$

5. **If $z(x+y)=x^2+y^2$, show that $\left(\frac{\partial z}{\partial x}-\frac{\partial z}{\partial y}\right)^2=4\left(1-\frac{\partial z}{\partial x}-\frac{\partial z}{\partial y}\right)$**

$z(x+y)=x^2+y^2$ ---- (1)

Differentiate (1) partially w. r. to x

$$\frac{\partial z}{\partial x}(x+y)+z(1+0)=2x+0$$

$$\frac{\partial z}{\partial x}=\frac{2x-z}{x+y}$$

Differentiate (1) partially w. r. to y

$$\frac{\partial z}{\partial y}(x+y)+z(0+1)=0+2y$$

$$\frac{\partial z}{\partial y}=\frac{2y-z}{x+y}$$

Therefore,

$$\left(\frac{\partial z}{\partial x}-\frac{\partial z}{\partial y}\right)^2=\left(\frac{2x-z}{x+y}-\frac{2y-z}{x+y}\right)^2$$

$$=\left(\frac{2x-z-2y+z}{x+y}\right)^2$$

$$=4\left(\frac{x-y}{x+y}\right)^2 \text{ ----- (2)}$$

$$4\left(1-\frac{\partial z}{\partial x}-\frac{\partial z}{\partial y}\right)=4\left(1-\frac{2x-z}{x+y}-\frac{2y-z}{x+y}\right)$$

$$=\frac{4}{x+y}(x+y-2x+z-2y+z)$$

$$=\frac{4}{x+y}(2z-x-y)$$

$$=\frac{4}{x+y}\left[2\left(\frac{x^2+y^2}{x+y}\right)-(x+y)\right]$$

$$=\frac{4}{(x+y)^2}(2x^2+2y^2-x^2-y^2-2xy)$$

$$=4\left(\frac{x^2+y^2-2xy}{(x+y)^2}\right)=4\left(\frac{x-y}{x+y}\right)^2 \text{ ----- (3)}$$

Equating (2) and (3),

$$\left(\frac{\partial z}{\partial x}-\frac{\partial z}{\partial y}\right)^2=4\left(1-\frac{\partial z}{\partial x}-\frac{\partial z}{\partial y}\right)$$

6. **If $v = (x^2 + y^2 + z^2)^{-\frac{1}{2}}$ prove that $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} = 0$**

$(x^2 + y^2 + z^2)^{\frac{1}{2}} = 1/v$

$\frac{1}{v^2} = x^2 + y^2 + z^2$

Differentiate partially w. r. to x

$-\frac{2}{v^3}\frac{\partial v}{\partial x} = 2x$

$\frac{\partial v}{\partial x} = -xv^3$

Differentiate again partially w. r. to x

$$\frac{\partial^2 v}{\partial x^2} = -v^3 - 3xv^2\frac{\partial v}{\partial x}$$
$$= -v^3 - 3xv^2(-xv^3)$$
$$= -v^3 + 3x^2v^5$$

Similarly,

$\frac{\partial^2 v}{\partial y^2} = -v^3 + 3y^2v^5$

$\frac{\partial^2 v}{\partial z^2} = -v^3 + 3z^2v^5$

Therefore,

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} = -3v^3 + 3(x^2 + y^2 + z^2)v^5$$
$$= -3v^3 + 3\frac{v^5}{v^2}$$
$$= -3v^3 + 3v^3 = 0$$

Home work:

7. If $u = x^2\tan^{-1}\frac{y}{x} - y^2\tan^{-1}\frac{x}{y}$, prove that $\frac{\partial^2 u}{\partial x \partial y} = \frac{x^2 - y^2}{x^2 + y^2}$.

8. If $u = \tan^{-1}\left(\frac{2xy}{x^2 - y^2}\right)$ prove that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$.

9. Find the first and second partial derivatives of $z = x^3 + y^3 - 3axy$

10. If $u = x^y$ show that $\frac{\partial^3 u}{\partial x^2 \partial y} = \frac{\partial^3 u}{\partial x \partial y \partial x}$

2.5 Total differentiation

Introduction:

- If $u = u(x, y)$ where $x = x(t)$ and $y = y(t)$ then $\frac{du}{dt} = \frac{\partial u}{\partial x}.\frac{dx}{dt} + \frac{\partial u}{\partial y}.\frac{dy}{dt}$
- If $u = u(x, y, z)$ where $x = x(t)$, $y = y(t)$ and $z = z(t)$ then

$$\frac{du}{dt} = \frac{\partial u}{\partial x}.\frac{dx}{dt} + \frac{\partial u}{\partial y}.\frac{dy}{dt} + \frac{\partial u}{\partial z}.\frac{dz}{dt}$$

Problems:

1. If $z = u^2 + v^2$ where $u = at^2$ and $v = 2at$ find $\frac{dz}{dt}$.

$$z = u^2 + v^2$$

$$\frac{\partial z}{\partial u} = 2u,\ \frac{\partial z}{\partial v} = 2v$$

$u = at^2$ $\frac{du}{dt} = 2at = v$	$v = 2at$ $\frac{dv}{dt} = 2a$

Therefore,

$$\frac{dz}{dt} = \frac{\partial z}{\partial u}.\frac{du}{dt} + \frac{\partial z}{\partial v}.\frac{dv}{dt}$$

$$= (2u)v + (2v)2a$$

$$= 2at^2(2at) + 4at(2a)$$

$$= 4a^2t(t^2 + 2)$$

2. If $u = xy^2 + x^2y$ with $x = at^2, y = 2at$ find $\frac{du}{dt}$ using partial derivatives.

$u = xy^2 + x^2y$

$\frac{\partial u}{\partial x} = y^2 + 2xy,\ \frac{\partial u}{\partial y} = 2xy + x^2$

$x = at^2$ $\frac{dx}{dt} = 2at = y$	$y = 2at$ $\frac{dy}{dt} = 2a$

Therefore, $\frac{du}{dt} = \frac{\partial u}{\partial x}.\frac{dx}{dt} + \frac{\partial u}{\partial y}.\frac{dy}{dt}$

$\frac{du}{dt} = (y^2 + 2xy)y + (2xy + x^2)2a$

$= y^3 + 2xy^2 + 4axy + 2ax^2$

$= (2at)^3 + 2(at^2)(2at)^2 + 4a\,(at^2)(2at) + 2a(at^2)^2$

$= 2a^3t^3(4 + 4t + 4 + t)$

$= 2a^3t^3(8 + 5t)$

3. If $u = tan^{-1}\left(\frac{y}{x}\right)$ where $x = e^t - e^{-t}$ and $y = e^t + e^{-t}$ find $\frac{du}{dt}$

$u = \tan^{-1}\frac{y}{x}$

$\frac{\partial u}{\partial x} = \frac{1}{1+\left(\frac{y}{x}\right)^2}\left(-\frac{y}{x^2}\right) = -\frac{y}{x^2+y^2}$

$\frac{\partial u}{\partial y} = \frac{1}{1+\left(\frac{y}{x}\right)^2}\left(\frac{1}{x}\right) = \frac{x}{x^2+y^2}$

$x = e^t - e^{-t}$ $\frac{dx}{dt} = e^t + e^{-t} = y$	$y = e^t + e^{-t}$ $\frac{dy}{dt} = e^t - e^{-t} = x$

Therefore, $\frac{du}{dt} = \frac{\partial u}{\partial x}.\frac{dx}{dt} + \frac{\partial u}{\partial y}.\frac{dy}{dt}$

$= \left(-\frac{y}{x^2+y^2}\right)y + \left(\frac{x}{x^2+y^2}\right)x$

$= \frac{x^2-y^2}{x^2+y^2}$

$= \frac{(e^t-e^{-t})^2-(e^t+e^{-t})^2}{(e^t-e^{-t})^2+(e^t+e^{-t})^2} = \frac{-2}{e^{2t}+e^{-2t}}$

4. **If $u = x^2 + y^2 + z^2$, where $x = e^{2t}, y = e^{2t}\cos 3t\,, z = e^{2t}\sin 3t$ find $\frac{du}{dt}$.**

$u = x^2 + y^2 + z^2$

$\frac{\partial u}{\partial x} = 2x,\ \ \frac{\partial u}{\partial y} = 2y,\ \ \frac{\partial u}{\partial z} = 2z$

$x = e^{2t}$ $\frac{dx}{dt} = 2e^{2t}$ $= 2x$	$y = e^{2t}\cos 3t$ $\frac{dy}{dt} = 2e^{2t}\cos 3t - 3e^{2t}\sin 3t$ $= 2y - 3z$	$z = e^{2t}\sin 3t$ $\frac{dz}{dt} = 2e^{2t}\sin 3t + 3e^{2t}\cos 3t$ $= 2z + 3y$

Therefore,

$$\frac{du}{dt} = \frac{\partial u}{\partial x}.\frac{dx}{dt} + \frac{\partial u}{\partial y}.\frac{dy}{dt} + \frac{\partial u}{\partial z}.\frac{dz}{dt}$$

$$= 2x(2x) + 2y(2y - 3z) + 2z(2z + 3y)$$

$$= 4(x^2 + y^2 + z^2)$$

$$= 4(e^{4t} + e^{4t})$$

$$= 8\,e^{4t}$$

5. **If $u = e^x \sin(yz)$, where $x = t^2, y = t - 1, z = \frac{1}{t}$ find $\frac{du}{dt}$ at $t = 1$.**

$u = e^x \sin(yz)$

$\frac{\partial u}{\partial x} = e^x \sin(yz),\ \frac{\partial u}{\partial y} = ze^x \cos(yz)\,, \frac{\partial u}{\partial z} = ye^x \cos(yz)$

$x = t^2$ $\frac{dx}{dt} = 2t$	$y = t - 1$ $\frac{dy}{dt} = 1$	$z = \frac{1}{t}$ $\frac{dz}{dt} = -\frac{1}{t^2}$

At $t = 1$,

$x = 1,\ y = 0,\ z = 1.$

$\frac{dx}{dt} = 2,\ \frac{dy}{dt} = 1,\ \frac{dz}{dt} = -1$

$\frac{\partial u}{\partial x} = 0,\ \frac{\partial u}{\partial y} = e,\ \frac{\partial u}{\partial z} = 0$

Therefore,

$$\frac{du}{dt} = \frac{\partial u}{\partial x}.\frac{dx}{dt} + \frac{\partial u}{\partial y}.\frac{dy}{dt} + \frac{\partial u}{\partial z}.\frac{dz}{dt}$$

$$= 0 + e.1 + 0$$

$$= e$$

6. If $u = x^2 + xy + y^2, x = t^2, y = 3t$ then find $\frac{du}{dt}$.

$u = x^2 + xy + y^2$

$\frac{\partial u}{\partial x} = 2x + y,\ \frac{\partial u}{\partial y} = x + 2y$

$x = t^2$ $\frac{dx}{dt} = 3t$	$y = 2t$ $\frac{dy}{dt} = 3$

Therefore,

$\frac{du}{dt} = \frac{\partial u}{\partial x}.\frac{dx}{dt} + \frac{\partial u}{\partial y}.\frac{dy}{dt}$

$= (2x + y)2t + (x + 2y)3$

$= (2t^2 + 3t)2t + (t^2 + 6t)3$

$= 4t^3 + 6t^2 + 3t^2 + 18t$

$= 4t^3 + 9t^2 + 18t$

7. If $u = x^3y^2 + x^2y^3$ with $x = at^2, y = 2at$ find $\frac{du}{dt}$ using partial derivatives.

$u = x^3y^2 + x^2y^3$

$\frac{\partial u}{\partial x} = 3x^2y^2 + 2xy^3,\ \frac{\partial u}{\partial y} = 2x^3y + 3x^2y^2$

$x = at^2$ $\frac{dx}{dt} = 2at = y$	$y = 2at$ $\frac{dy}{dt} = 2a$

Therefore,

$\frac{du}{dt} = \frac{\partial u}{\partial x}.\frac{dx}{dt} + \frac{\partial u}{\partial y}.\frac{dy}{dt}$

$= (3x^2y^2 + 2xy^3)y + (2x^3y + 3x^2y^2)2a$

$= 3x^2y^3 + 2xy^4 + 4ax^3y + 6ax^2y^2$

$= 3a^2t^4.8a^3t^3 + 2at^2.16a^4t^4 + 4a.a^3t^6.2at + 6a.a^2t^4.4a^2t^2$

$= 8a^5t^6(3t + 4 + t + 3)$

$= 8a^5t^6(4t + 7)$

8. If $\boldsymbol{u = \sin\frac{x}{y}}$, where $\boldsymbol{x = e^t, y = e^{t^2}}$ find $\boldsymbol{\frac{du}{dt}}$.

$u = \sin\frac{x}{y}$

$\frac{\partial u}{\partial x} = \frac{1}{y}\cos\frac{x}{y}, \ \frac{\partial u}{\partial y} = -\frac{x}{y^2}\cos\frac{x}{y}$

$x = e^t, \ \frac{dx}{dt} = e^t = x$

$y = e^{t^2}, \ \frac{dy}{dt} = 2te^{t^2} = 2ty$

Therefore,

$\frac{du}{dt} = \frac{\partial u}{\partial x}.\frac{dx}{dt} + \frac{\partial u}{\partial y}.\frac{dy}{dt}$

$= \left(\frac{1}{y}\cos\frac{x}{y}\right)x + \left(-\frac{x}{y^2}\cos\frac{x}{y}\right)2ty$

$= (1-2t)\frac{x}{y}\cos\frac{x}{y}$

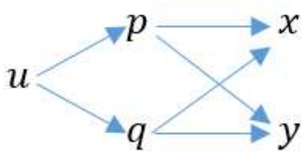
$= (1-2t)\left(e^{t-t^2}\cos e^{t-t^2}\right)$

2.6 Partial derivatives of composite functions

Introduction

- If $u = f(p,q)$ where $p = p(x,y)$and $q = q(x,y)$ then z is a composite function of x and y. Partial derivatives of composite function z are given by

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial p}\frac{\partial p}{\partial x} + \frac{\partial u}{\partial q}\frac{\partial q}{\partial x}$$
$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial p}\frac{\partial p}{\partial y} + \frac{\partial u}{\partial q}\frac{\partial q}{\partial y}$$

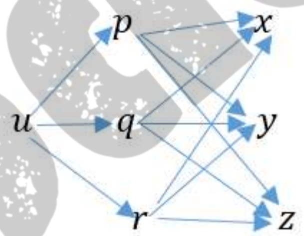


- If $u = f(p,q,r)$ where $p = p(x,y)$, $q = q(x,y)$ and $r = r(x,y)$ then z is a composite function of x and y. Partial derivatives of composite function z are given by

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial p}\frac{\partial p}{\partial x} + \frac{\partial u}{\partial q}\frac{\partial q}{\partial x} + \frac{\partial u}{\partial r}\frac{\partial r}{\partial x},$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial p}\frac{\partial p}{\partial y} + \frac{\partial u}{\partial q}\frac{\partial q}{\partial y} + \frac{\partial u}{\partial r}\frac{\partial r}{\partial y}$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial p}\frac{\partial p}{\partial z} + \frac{\partial u}{\partial q}\frac{\partial q}{\partial z} + \frac{\partial u}{\partial r}\frac{\partial r}{\partial z}$$



Problems:

1. If $z = f(x,y)$, where $x = e^{u} + e^{-v}$, $y = e^{-u} - e^{v}$ then P.T. $\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}$.

$x = e^{u} + e^{-v}$	$y = e^{-u} - e^{v}$
$\frac{\partial x}{\partial u} = e^{u}$	$\frac{\partial y}{\partial u} = -e^{-u}$
$\frac{\partial x}{\partial v} = -e^{-v}$	$\frac{\partial y}{\partial v} = -e^{v}$

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} = = \frac{\partial z}{\partial x}(e^{u}) + \frac{\partial z}{\partial y}(-e^{-u}) \quad \text{------ (1)}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v} = \frac{\partial z}{\partial x}(-e^{-v}) + \frac{\partial z}{\partial y}(-e^{v}) \quad \text{------ (2)}$$

$(1) - (2)$ gives,

$$\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}(e^{u} + e^{-v}) + \frac{\partial z}{\partial y}(-e^{-u} + e^{v})$$

$$= (e^{u} + e^{-v})\frac{\partial z}{\partial x} - (e^{-u} - e^{v})\frac{\partial z}{\partial y}$$

$$= x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}$$

2. If $z = f(x, y)$, where $x = e^u \cos v$, $y = e^u \sin v$ then P.T. $y\frac{\partial z}{\partial u} + x\frac{\partial z}{\partial v} = e^{2u}\frac{\partial z}{\partial y}$.

$x = e^u \cos v$ $\frac{\partial x}{\partial u} = e^u \cos v = x$ $\frac{\partial x}{\partial v} = -e^u \sin v = -y$	$y = e^u \sin v$ $\frac{\partial y}{\partial u} = e^u \sin v = y$ $\frac{\partial y}{\partial v} = e^u \cos v = x$

$$y\frac{\partial z}{\partial u} = y\left\{\frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}\right\}$$

$$= y\left\{\frac{\partial z}{\partial x}(x) + \frac{\partial z}{\partial y}(y)\right\}$$

$$= xy\frac{\partial z}{\partial x} + y^2\frac{\partial z}{\partial y} \qquad \text{---- (1)}$$

$$x\frac{\partial z}{\partial v} = x\left\{\frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}\right\}$$

$$= x\left\{\frac{\partial z}{\partial x}(-y) + \frac{\partial z}{\partial y}(x)\right\}$$

$$= -xy\frac{\partial z}{\partial x} + x^2\frac{\partial z}{\partial y} \qquad \text{---- (2)}$$

(1) +(2) gives,

$$y\frac{\partial z}{\partial u} + x\frac{\partial z}{\partial v} = xy\frac{\partial z}{\partial x} + y^2\frac{\partial z}{\partial y} - xy\frac{\partial z}{\partial x} + x^2\frac{\partial z}{\partial y}$$

$$= (x^2 + y^2)\frac{\partial z}{\partial y}$$

$$= (e^{2u}\cos^2 u + e^{2u}\sin^2 v)\frac{\partial z}{\partial y}$$

$$= e^{2u}\frac{\partial z}{\partial y}$$

3. If $u = f(x - y, y - z, z - x)$ then prove that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$.

$p = x - y$ $\frac{\partial p}{\partial x} = 1$ $\frac{\partial p}{\partial y} = -1$ $\frac{\partial p}{\partial z} = 0$	$q = y - z$ $\frac{\partial q}{\partial x} = 0$ $\frac{\partial q}{\partial y} = 1$ $\frac{\partial q}{\partial z} = -1$	$r = z - x$ $\frac{\partial r}{\partial x} = -1$ $\frac{\partial r}{\partial y} = 0$ $\frac{\partial r}{\partial z} = 1$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial p}\frac{\partial p}{\partial x} + \frac{\partial u}{\partial q}\frac{\partial q}{\partial x} + \frac{\partial u}{\partial r}\frac{\partial r}{\partial x}$$

$$= \frac{\partial u}{\partial p}(1) + \frac{\partial u}{\partial q}(0) + \frac{\partial u}{\partial r}(-1)$$

$$= \frac{\partial u}{\partial p} - \frac{\partial u}{\partial r} \qquad \text{---- (1)}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial p}\frac{\partial p}{\partial y} + \frac{\partial u}{\partial q}\frac{\partial q}{\partial y} + \frac{\partial u}{\partial r}\frac{\partial r}{\partial y}$$

$$= \frac{\partial u}{\partial p}(-1) + \frac{\partial u}{\partial q}(1) + \frac{\partial u}{\partial r}(0)$$

$$= \frac{\partial u}{\partial q} - \frac{\partial u}{\partial p} \qquad \text{---- (2)}$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial p}\frac{\partial p}{\partial z} + \frac{\partial u}{\partial q}\frac{\partial q}{\partial z} + \frac{\partial u}{\partial r}\frac{\partial r}{\partial z}$$

$$= \frac{\partial u}{\partial p}(0) + \frac{\partial u}{\partial q}(-1) + \frac{\partial u}{\partial r}(1)$$

$$= \frac{\partial u}{\partial r} - \frac{\partial u}{\partial q} \qquad \text{---- (3)}$$

(1) + (2) + (3) gives,

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{\partial u}{\partial p} - \frac{\partial u}{\partial q} + \frac{\partial u}{\partial q} - \frac{\partial u}{\partial r} + \frac{\partial u}{\partial r} - \frac{\partial u}{\partial p}$$

$$= 0$$

4. If $u = f(2x - 3y\,, 3y - 4z\,, 4z - 2x)$ then prove that $\frac{1}{2}\frac{\partial u}{\partial x} + \frac{1}{3}\frac{\partial u}{\partial y} + \frac{1}{4}\frac{\partial u}{\partial z} = 0$.

$p = 2x - 3y$	$q = 3y - 4z$	$r = 4z - 2x$
$\frac{\partial p}{\partial x} = 2$	$\frac{\partial q}{\partial x} = 0$	$\frac{\partial r}{\partial x} = -2$
$\frac{\partial p}{\partial y} = -3$	$\frac{\partial q}{\partial y} = 3$	$\frac{\partial r}{\partial y} = 0$
$\frac{\partial p}{\partial z} = 0$	$\frac{\partial q}{\partial z} = -4$	$\frac{\partial r}{\partial z} = 4$

Therefore,

$$\frac{1}{2}\frac{\partial u}{\partial x} = \frac{1}{2}\left\{\frac{\partial u}{\partial p}\frac{\partial p}{\partial x} + \frac{\partial u}{\partial q}\frac{\partial q}{\partial x} + \frac{\partial u}{\partial r}\frac{\partial r}{\partial x}\right\}$$

$$= \frac{1}{2}\left\{\frac{\partial u}{\partial p}(2) + 0 + \frac{\partial u}{\partial r}(-2)\right\}$$

$$= \frac{\partial u}{\partial p} - \frac{\partial u}{\partial r}$$

$$\frac{1}{3}\frac{\partial u}{\partial y} = \frac{1}{3}\left\{\frac{\partial u}{\partial p}\frac{\partial p}{\partial y} + \frac{\partial u}{\partial q}\frac{\partial q}{\partial y} + \frac{\partial u}{\partial r}\frac{\partial r}{\partial y}\right\}$$

$$= \frac{1}{3}\left\{\frac{\partial u}{\partial p}(-3) + \frac{\partial u}{\partial q}(3) + \right\}0$$

$$= \frac{\partial u}{\partial q} - \frac{\partial u}{\partial p}$$

$$\frac{1}{4}\frac{\partial u}{\partial z} = \frac{1}{4}\left\{\frac{\partial u}{\partial p}\frac{\partial p}{\partial z} + \frac{\partial u}{\partial q}\frac{\partial q}{\partial z} + \frac{\partial u}{\partial r}\frac{\partial r}{\partial z}\right\}$$

$$= \frac{1}{4}\left\{0 + \frac{\partial u}{\partial q}(-4) + \frac{\partial u}{\partial r}(4)\right\}$$

$$= \frac{\partial u}{\partial r} - \frac{\partial u}{\partial q}$$

(1) + (2) + (3) gives,

$$\frac{1}{2}\frac{\partial u}{\partial x} + \frac{1}{3}\frac{\partial u}{\partial y} + \frac{1}{4}\frac{\partial u}{\partial z} = \frac{\partial u}{\partial p} - \frac{\partial u}{\partial r} + \frac{\partial u}{\partial q} - \frac{\partial u}{\partial p} + \frac{\partial u}{\partial r} - \frac{\partial u}{\partial q}$$

$$= 0$$

5. If $u = f\left(\frac{x}{y}, \frac{y}{z}, \frac{z}{x}\right)$ then prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$. **(May 22)**

$p = \frac{x}{y}$	$q = \frac{y}{z}$	$r = \frac{z}{x}$
$\frac{\partial p}{\partial x} = \frac{1}{y}$	$\frac{\partial q}{\partial x} = 0$	$\frac{\partial r}{\partial x} = -\frac{z}{x^2}$
$\frac{\partial p}{\partial y} = -\frac{x}{y^2}$	$\frac{\partial q}{\partial y} = \frac{1}{z}$	$\frac{\partial r}{\partial y} = 0$
$\frac{\partial p}{\partial z} = 0$	$\frac{\partial q}{\partial z} = -\frac{y}{z^2}$	$\frac{\partial r}{\partial z} = \frac{1}{x}$

$$x\frac{\partial u}{\partial x} = x\left\{\frac{\partial u}{\partial p}\frac{\partial p}{\partial x} + \frac{\partial u}{\partial q}\frac{\partial q}{\partial x} + \frac{\partial u}{\partial r}\frac{\partial r}{\partial x}\right\}$$

$$= x\left\{\frac{\partial u}{\partial p}\left(\frac{1}{y}\right) + 0 + \frac{\partial u}{\partial r}\left(-\frac{z}{x^2}\right)\right\}$$

$$= \frac{x}{y}\frac{\partial u}{\partial p} - \frac{z}{x}\frac{\partial u}{\partial r} \qquad \text{---- (1)}$$

$$y\frac{\partial u}{\partial y} = y\left\{\frac{\partial u}{\partial p}\frac{\partial p}{\partial y} + \frac{\partial u}{\partial q}\frac{\partial q}{\partial y} + \frac{\partial u}{\partial r}\frac{\partial r}{\partial y}\right\}$$

$$= y\left\{\frac{\partial u}{\partial p}\left(-\frac{x}{y^2}\right) + \frac{\partial u}{\partial q}\left(\frac{1}{z}\right) + 0\right\}$$

$$= \frac{y}{z}\frac{\partial u}{\partial q} - \frac{x}{y}\frac{\partial u}{\partial p} \qquad \text{---- (2)}$$

$$z\frac{\partial u}{\partial z} = z\left\{\frac{\partial u}{\partial p}\frac{\partial p}{\partial z} + \frac{\partial u}{\partial q}\frac{\partial q}{\partial z} + \frac{\partial u}{\partial r}\frac{\partial r}{\partial z}\right\}$$

$$= z\left\{0 + \frac{\partial u}{\partial q}\left(-\frac{y}{z^2}\right) + \frac{\partial u}{\partial r}\left(\frac{1}{x}\right)\right\}$$

$$= \frac{z}{x}\frac{\partial u}{\partial r} - \frac{y}{z}\frac{\partial u}{\partial q} \qquad \text{----- (3)}$$

(1) + (2) + (3) gives,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = \frac{x}{y}\frac{\partial u}{\partial p} - \frac{z}{x}\frac{\partial u}{\partial r} + \frac{y}{z}\frac{\partial u}{\partial q} - \frac{x}{y}\frac{\partial u}{\partial p} + \frac{z}{x}\frac{\partial u}{\partial r} - \frac{y}{z}\frac{\partial u}{\partial q}$$

$$= 0$$

6. If $u = f\left(\frac{y-x}{xy}, \frac{z-x}{zx}\right)$ then prove that $x^2\frac{\partial u}{\partial x} + y^2\frac{\partial u}{\partial y} + z^2\frac{\partial u}{\partial z} = 0$.

$p = \frac{y-x}{xy} = \frac{1}{x} - \frac{1}{y}$	$q = \frac{z-x}{zx} = \frac{1}{x} - \frac{1}{z}$
$\frac{\partial p}{\partial x} = -\frac{1}{x^2}$	$\frac{\partial q}{\partial x} = -\frac{1}{x^2}$
$\frac{\partial p}{\partial y} = \frac{1}{y^2}$	$\frac{\partial q}{\partial y} = 0$
$\frac{\partial p}{\partial z} = 0$	$\frac{\partial q}{\partial z} = \frac{1}{z^2}$

Therefore,

$$x^2\frac{\partial u}{\partial x} = x^2\left\{\frac{\partial u}{\partial p}\frac{\partial p}{\partial x} + \frac{\partial u}{\partial q}\frac{\partial q}{\partial x}\right\}$$

$$= x^2\left\{\frac{\partial u}{\partial p}\left(-\frac{1}{x^2}\right) + \frac{\partial u}{\partial q}\left(-\frac{1}{x^2}\right)\right\}$$

$$= -\frac{\partial u}{\partial p} - \frac{\partial u}{\partial q} \qquad \text{---- (1)}$$

$$y^2\frac{\partial u}{\partial y} = y^2\left\{\frac{\partial u}{\partial p}\frac{\partial p}{\partial y} + \frac{\partial u}{\partial q}\frac{\partial q}{\partial y}\right\}$$

$$= y^2\left\{\frac{\partial u}{\partial p}\left(\frac{1}{y^2}\right) + 0\right\}$$

$$= \frac{\partial u}{\partial p} \qquad \text{---- (2)}$$

$$z^2\frac{\partial u}{\partial z} = z^2\left\{\frac{\partial u}{\partial p}\frac{\partial p}{\partial z} + \frac{\partial u}{\partial q}\frac{\partial q}{\partial z}\right\}$$

$$= z^2\left\{0 + \frac{\partial u}{\partial q}\left(\frac{1}{z^2}\right)\right\}$$

$$= \frac{\partial u}{\partial q} \qquad \text{---- (3)}$$

(1) + (2) + (3) gives,

$$x^2\frac{\partial u}{\partial x} + y^2\frac{\partial u}{\partial y} + z^2\frac{\partial u}{\partial z} = -\frac{\partial u}{\partial p} - \frac{\partial u}{\partial q} + \frac{\partial u}{\partial p} + \frac{\partial u}{\partial q} = 0$$

7. If $z = f(x, y), x = r\ \cos\theta\ , y = r\ \ \sin\theta$ then P.T. $\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = \left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2$

$x = r\cos\theta$ $\frac{\partial x}{\partial r} = \cos\theta = \frac{x}{r}$ $\frac{\partial x}{\partial \theta} = -r\sin\theta = -y$	$y = r\sin\theta$ $\frac{\partial y}{\partial r} = \sin\theta = \frac{y}{r}$ $\frac{\partial y}{\partial \theta} = r\cos\theta = x$

$$\left(\frac{\partial z}{\partial r}\right)^2 = \left(\frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r}\right)^2$$

$$= \left\{\frac{\partial z}{\partial x}\left(\frac{x}{r}\right) + \frac{\partial z}{\partial y}\left(\frac{y}{r}\right)\right\}^2$$

$$= \frac{1}{r^2}\left\{x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}\right\}^2 \qquad \text{----- (1)}$$

$$\frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2 = \frac{1}{r^2}\left\{\frac{\partial z}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial \theta}\right\}^2$$

$$= \frac{1}{r^2}\left\{-y\frac{\partial z}{\partial x} + x\frac{\partial z}{\partial y}\right\}^2 \qquad \text{----- (2)}$$

$(1) + (2)$ gives,

$$\left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2 = \frac{1}{r^2}\left\{(x^2 + y^2)\left(\frac{\partial z}{\partial x}\right)^2 + (x^2 + y^2)\left(\frac{\partial z}{\partial y}\right)^2\right\}$$

$$= \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 \qquad [\because x^2 + y^2 = r^2]$$

Home work:

8. If $z = f(u, v), u = x^2 - y^2, v = 2xy$ then prove that

$$\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = 4(x^2 + y^2)\left[\left(\frac{\partial z}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2\right]$$

9. If $u = f\left(xz, \frac{y}{z}\right)$ then prove that $x\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y} - z\frac{\partial u}{\partial z} = 0$.

10. If $u = f\left(\frac{x}{z}, \frac{y}{z}\right)$ then prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$.

2.6 Jacobians

Introduction:

- If u and v are functions of two independent variables x and y then

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \quad \text{(OR)} \quad J\left(\frac{u,v}{x,y}\right) = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$$

- If u, v and w are functions of three independent variables x, y and z then

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} \quad \text{(OR)} \quad J\left(\frac{u,v}{x,y}\right) = \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix}$$

Problems:

1. **If $\boldsymbol{x = r\cos\theta}$, $\boldsymbol{y = r\sin\theta}$ then prove that $\boldsymbol{\frac{\partial(x,y)}{\partial(r,\theta)} = r}$.**

$x = r\cos\theta$	$y = r\sin\theta$
$\frac{\partial x}{\partial r} = \cos\theta$	$\frac{\partial y}{\partial r} = \sin\theta$
$\frac{\partial x}{\partial \theta} = -r\sin\theta$	$\frac{\partial y}{\partial \theta} = r\cos\theta$

Therefore, $\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} \end{vmatrix}$

$$= \begin{vmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\, sun^2\theta = r$$

2. **If $\boldsymbol{x = u(1-v), y = uv}$ then find $\boldsymbol{\frac{\partial(x,y)}{\partial(u,v)}}$.**

$x = u - uv$	$y = uv$
$\frac{\partial x}{\partial u} = 1 - v$	$\frac{\partial y}{\partial u} = v$
$\frac{\partial x}{\partial v} = -u$	$\frac{\partial y}{\partial v} = u$

Therefore, $\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix}$

$$= \begin{vmatrix} 1-v & v \\ -u & u \end{vmatrix} = u - uv + uv = u$$

3. **If $x = u(1-v), y = uv$ then find $\frac{\partial(u,v)}{\partial(x,y)}$.**

$u = x + y$ $\frac{\partial u}{\partial x} = 1$ $\frac{\partial u}{\partial y} = 1$	$v = \frac{y}{x+y}$ $\frac{\partial v}{\partial x} = -\frac{y}{(x+y)^2}$ $\frac{\partial u}{\partial y} = \frac{x+y-y}{(x+y)^2} = \frac{x}{(x+y)^2}$

Therefore, $\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial v} & \frac{\partial v}{\partial y} \end{vmatrix}$

$$= \begin{vmatrix} 1 & -\frac{y}{(x+y)^2} \\ 1 & \frac{x}{(x+y)^2} \end{vmatrix}$$

$$= \frac{x}{(x+y)^2} + \frac{y}{(x+y)^2}$$

$$= \frac{1}{x+y}$$

4. **If $u = x^2 + y^2 + z^2, v = xy + yz + zx, w = x + y + z$ then prove that u, v *and* w are functionally dependent.**

$u = x^2 + y^2 + z^2$ $\frac{\partial u}{\partial x} = 2x$ $\frac{\partial u}{\partial y} = 2y$ $\frac{\partial u}{\partial y} = 2y$	$v = xy + yz + zx$ $\frac{\partial v}{\partial x} = y + z$ $\frac{\partial v}{\partial y} = x + z$ $\frac{\partial v}{\partial y} = x + y$	$w = x + y + z$ $\frac{\partial w}{\partial x} = 1$ $\frac{\partial w}{\partial y} = 1$ $\frac{\partial w}{\partial y} = 1$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} u_x & v_x & w_x \\ u_y & v_y & w_y \\ u_z & v_z & w_z \end{vmatrix}$$

$$= \begin{vmatrix} 2x & y+z & 1 \\ 2y & z+x & 1 \\ 2z & x+y & 1 \end{vmatrix}$$

$$= 2\begin{vmatrix} x & y+z & 1 \\ y-x & x-y & 0 \\ z-x & x-z & 0 \end{vmatrix}$$

$$= 2(x-y)(x-z)\begin{vmatrix} x & y+z & 1 \\ -1 & 1 & 0 \\ -1 & 1 & 0 \end{vmatrix}$$

$$= 0$$

Therefore, u, v *and* w are functionally dependent.

5. **If $u = x^2 + 3y^2 - z^3, v = 4x^2yz, w = 2z^2 - xy$, evaluate $\frac{\partial(u,v,w)}{\partial(x,y,z)}$ at $(1,-1,\ 0)$.**

(May 22)

$u = x + 3y^2 - z^3$ $\frac{\partial u}{\partial x} = 1$ $\frac{\partial u}{\partial y} = 6y$ $\frac{\partial u}{\partial z} = -3z^2$	$v = 4x^2yz$ $\frac{\partial v}{\partial x} = 8xyz$ $\frac{\partial v}{\partial y} = 4x^2z$ $\frac{\partial v}{\partial z} = 4x^2y$	$w = 2z^2 - xy$ $\frac{\partial w}{\partial x} = -y$ $\frac{\partial w}{\partial y} = -x$ $\frac{\partial w}{\partial z} = 4z$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} u_x & v_x & w_x \\ u_y & v_y & w_y \\ u_z & v_z & w_z \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 8xyz & -y \\ 6y & 4x^2z & -x \\ -3z^2 & 4x^2y & 4z \end{vmatrix}$$

At $(1,-1,0)$,

$$= \begin{vmatrix} 1 & 0 & 1 \\ -6 & 0 & -1 \\ 0 & -4 & 0 \end{vmatrix} = 20$$

6. **If $u = \frac{xy}{z}, v = \frac{yz}{x}, w = \frac{zx}{y}$ find $J\left(\frac{u,v,w}{x,y,z}\right)$** [Jan 17

$u = \frac{xy}{z}$ $\frac{\partial u}{\partial x} = \frac{y}{z}$ $\frac{\partial u}{\partial y} = \frac{x}{z}$ $\frac{\partial u}{\partial z} = -\frac{xy}{z^2}$	$v = \frac{yz}{x}$ $\frac{\partial v}{\partial x} = -\frac{yz}{x^2}$ $\frac{\partial v}{\partial y} = \frac{z}{x}$ $\frac{\partial v}{\partial z} = \frac{y}{x}$	$w = \frac{zx}{y}$ $\frac{\partial w}{\partial x} = \frac{z}{y}$ $\frac{\partial w}{\partial y} = -\frac{zx}{y^2}$ $\frac{\partial w}{\partial z} = \frac{x}{y}$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} u_x & v_x & w_x \\ u_y & v_y & w_y \\ u_z & v_z & w_z \end{vmatrix}$$

$$= \begin{vmatrix} \frac{y}{z} & -\frac{yz}{x^2} & \frac{z}{y} \\ \frac{x}{z} & \frac{z}{x} & -\frac{zx}{y^2} \\ -\frac{xy}{z^2} & \frac{y}{x} & \frac{x}{y} \end{vmatrix}$$

$$= \frac{1}{x^2y^2z^2}\begin{vmatrix} yz & -yz & yz \\ zx & -zx & -zx \\ -xy & xy & xy \end{vmatrix}$$

$$= \frac{x^2y^2z^2}{x^2y^2z^2}\begin{vmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & 1 \end{vmatrix} = 4$$

7. If $x = r\sin\theta\cos\phi, y = r\sin\theta\sin\phi, z = r\cos\theta$ find $J\left(\frac{x,y,z}{r,\theta,\phi}\right)$ [July 16

$x = r\sin\theta\cos\phi$ $\frac{\partial x}{\partial r} = \sin\theta\cos\phi$ $\frac{\partial x}{\partial \theta} = r\cos\theta\cos\phi$ $\frac{\partial x}{\partial \phi} = -r\sin\theta\sin\phi$	$y = r\sin\theta\sin\phi$ $\frac{\partial y}{\partial r} = sin\theta\sin\phi$ $\frac{\partial y}{\partial \theta} = r\cos\theta\sin\phi$ $\frac{\partial y}{\partial \phi} = r\sin\theta\cos\phi$	$z = r\cos\theta$ $\frac{\partial z}{\partial r} = \cos\theta$ $\frac{\partial z}{\partial \theta} = -r\sin\theta$ $\frac{\partial z}{\partial \phi} = 0$

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \begin{vmatrix} x_r & y_r & z_r \\ x_\theta & y_\theta & z_\theta \\ x_\phi & y_\phi & z_\phi \end{vmatrix}$$

$$= \begin{vmatrix} \sin\theta\cos\phi & sin\theta\sin\phi & \cos\theta \\ r\cos\theta\cos\phi & r\cos\theta\sin\phi & 0 \\ -r\sin\theta\sin\phi & r\sin\theta\cos\phi & -r\sin\theta \end{vmatrix}$$

$$= r^2\sin\theta \begin{vmatrix} \sin\theta\cos\phi & sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{vmatrix}$$

$$= r^2\sin\theta\{\cos\theta(\cos\theta) + \sin\theta(\sin\theta) - 1(0)\}$$

$$= r^2\sin\theta$$

8. If $u = x + y + z, uv = y + z, uvw = z$ find $\frac{\partial(x,y,z)}{\partial(u,v,w)}$. [Jan 16,

$x = u - uv$ $\frac{\partial x}{\partial u} = 1 - v$ $\frac{\partial x}{\partial v} = -u$ $\frac{\partial x}{\partial w} = 0$	$y = uv - uvw$ $\frac{\partial y}{\partial u} = v - vw$ $\frac{\partial y}{\partial v} = u - uw$ $\frac{\partial y}{\partial w} = -uv$	$z = uvw$ $\frac{\partial z}{\partial u} = vw$ $\frac{\partial z}{\partial v} = uw$ $\frac{\partial z}{\partial w} = uv$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} x_u & y_u & z_u \\ x_v & y_v & z_v \\ x_w & y_w & z_w \end{vmatrix}$$

$$= \begin{vmatrix} 1-v & v-vw & vw \\ -u & u-uw & uw \\ 0 & -uv & uv \end{vmatrix}$$

$$= u^2v \begin{vmatrix} 1-v & v-vw & vw \\ -1 & 1-w & w \\ 0 & -1 & 1 \end{vmatrix}$$

$$= u^2v \begin{vmatrix} 1-v & v & vw \\ -1 & 1 & w \\ 0 & 0 & 1 \end{vmatrix}, c_2 \to c_2 + c_3$$

$$= u^2v(1 - v + v)$$

$$= u^2v$$

9. If $x = e^u \cos v\, , y = e^u \sin v$ then find $\frac{\partial(x,y)}{\partial(u,v)}$.

$x = e^u \cos v$	$y = e^u \sin v$
$\frac{\partial x}{\partial u} = e^u \cos v$	$\frac{\partial y}{\partial u} = e^u \sin v$
$\frac{\partial x}{\partial v} = -e^u \sin v$	$\frac{\partial y}{\partial v} = e^u \cos v$

Therefore, $\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix}$

$$= \begin{vmatrix} e^u \cos v & e^u \sin v \\ -e^u \sin v & e^u \cos v \end{vmatrix}$$

$$= e^{2u} \begin{vmatrix} \cos v & \sin v \\ -\sin v & \cos v \end{vmatrix}$$

$$= e^{2u}$$

10. If $u = x^2 - y^2, v = 2xy$ and $x = r\cos\theta\, , y = r\sin\theta$ find $\frac{\partial(u,v)}{\partial(r,\theta)}$

$u = x^2 - y^2$	$v = 2xy$		$x = r\cos\theta$	$y = r\sin\theta$
$\frac{\partial u}{\partial x} = 2x$	$\frac{\partial v}{\partial x} = 2y$		$\frac{\partial x}{\partial r} = \cos\theta$	$\frac{\partial y}{\partial r} = \sin\theta$
$\frac{\partial u}{\partial y} = -2y$	$\frac{\partial v}{\partial y} = 2x$		$\frac{\partial x}{\partial \theta} = -r\sin\theta$	$\frac{\partial y}{\partial \theta} = r\cos\theta$

$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial v} & \frac{\partial v}{\partial y} \end{vmatrix}$

$= \begin{vmatrix} 2x & 2y \\ -2y & 2x \end{vmatrix}$

$= 4(x^2 + y^2) = 4r^2$

$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} \end{vmatrix}$

$= \begin{vmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{vmatrix}$

$= r\cos^2\theta + r\sin^2\theta = r$

Therefore, $\frac{\partial(u,v)}{\partial(r,\theta)} = \frac{\partial(u,v)}{\partial(x,y)} \times \frac{\partial(x,y)}{\partial(r,\theta)} = 4r^2 \times r = 4r^3$

Home work:

11. If $u = \frac{x+y}{1-xy}$ and $v = \tan^{-1} x + \tan^{-1} y$ find $\frac{\partial(x,y)}{\partial(u,v)}$. Ans: 0

12. If $x = \rho\cos\phi\, , y = \rho\sin\phi\, , z = z$ show that $\frac{\partial(x,y,z)}{\partial(\rho,\phi,z)} = \rho$

2.8 Maxima and minima for a function of two variables

Introduction:

- **Extreme point:**

  A point at which $f(x, y)$ attains maximum or minimum.

- **Saddle point:**

  A point at which $f(x, y)$ attains neither max. nor minimum.

- **Necessary conditions:**

  The necessary conditions for $f(x, y)$ to have a max. or min. at (a, b) are

$$\left(\frac{\partial f}{\partial x}\right)_{(a,b)} = 0 \text{ and } \left(\frac{\partial f}{\partial y}\right)_{(a,b)} = 0\ .$$

- **Notation:**

$$p = \frac{\partial f}{\partial x}\ ,\ q = \frac{\partial f}{\partial y}\ , r = \frac{\partial^2 f}{\partial x^2}\ , t = \frac{\partial^2 f}{\partial y^2}\ , s = \frac{\partial^2 f}{\partial x \partial y}$$

- **Sufficient conditions:**

  The sufficient conditions for $f(x, y)$ to have

  (i) Max. at (a, b) is that $rt - s^2 > 0$ and $r < 0$.

  (ii) Min. at (a, b) is that $rt - s^2 > 0$ and $r > 0$.

Working rule:

- Find critical points by solving $p = 0$ and $q = 0$.
- Find $rt - s^2$ and r at each critical point.
- Write the conclusion using the following table:

At (a, b), if	$f(x, y)$ attains
$rt - s^2 > 0\ , r < 0$	Maximum
$rt - s^2 > 0\ , r > 0$	Minimum
$rt - s^2 < 0$	Neither maximum nor minimum
$rt - s^2 = 0$	Saddle point

1. **Show that $f(x,y) = xy(a-x-y), a > 0$ is maximum at the point $\left(\frac{a}{3}, \frac{a}{3}\right)$.**

$\frac{\partial f}{\partial x} = y(a-x-y) - xy = y(a-2x-y))$

$\frac{\partial f}{\partial y} = x(a-x-y) - xy = x(a-x-2y))$

At $\left(\frac{a}{3}, \frac{a}{3}\right), p = 0$ and $q = 0$.

Therefore, $\left(\frac{a}{3}, \frac{a}{3}\right)$ is an extremum point.

$r = \frac{\partial^2 f}{\partial x^2} = -2y$

$t = \frac{\partial^2 f}{\partial y^2} = -2x$

$s = \frac{\partial^2 f}{\partial x \partial y} = a - 2x - 2y$

$rt - s^2 = 4xy - (a-2x-2y)^2$

At $\left(\frac{a}{3}, \frac{a}{3}\right), r < 0$ and $rt - s^2 = \frac{4a^2}{9} - \frac{a^2}{9} = \frac{a^2}{3}$

2. **Find the extreme values of the function $f(x,y) = x^3 + y^3 - 3axy$.**

$f(x,y) = x^3 + y^3 - 3axy$

$\frac{\partial f}{\partial x} = 3x^2 - 3ay, \ \frac{\partial f}{\partial y} = 3y^2 - 3ax$

$\frac{\partial^2 f}{\partial x^2} = 6x, \ \frac{\partial^2 f}{\partial y^2} = 6y, \ \frac{\partial^2 f}{\partial x \partial y} = -3a$

Step:1 Find critical points

$p = \frac{\partial f}{\partial x} = 0 \Rightarrow x^2 = ay$ ---- (1)

$q = \frac{\partial f}{\partial y} = 0 \Rightarrow y^2 = ax$ ---- (2)

$(1) \times x - (2) \times y \Rightarrow x^3 = y^3 \Rightarrow x = y$

Put $y = x$ in (1). We get Critical points $(0,0), (a,a)$.

Step:2 Find $rt - s^2$ at each critical point

$r = 6x, \ t = 6y, \ s = -3a, \ rt - s^2 = 36xy - 9a^2$

Critical points	$rt - s^2 = 36xy - 9a^2$	$r = 6x$	Remark
$(0,0)$ (a,a)	0 $36a^2 - 9a^2$, Positive.	---- Positive	Saddle point Minimum

Step:3 Conclusion

$f(x,y)$ attains minimum at (a,a).

Minimum value $= f(a,a) = a^3 + a^3 - 3a^3 = -a^3$.

3. **Find the extreme values of the function $f(x,y) = x^3 + 3x^2 + 4xy + y^2$.**

$f(x,y) = x^3 + 3x^2 + 4xy + y^2$

$\frac{\partial f}{\partial x} = 3x^2 + 6x + 4y,\ \frac{\partial f}{\partial y} = 4x + 2y$

$\frac{\partial^2 f}{\partial x^2} = 6x + 6,\ \ \frac{\partial^2 f}{\partial y^2} = 2,\ \ \frac{\partial^2 f}{\partial x \partial y} = 4$

Step:1 Find critical points

$p = \frac{\partial f}{\partial x} = 0 \Rightarrow 3x^2 + 6x + 4y = 0$---- (1)

$q = \frac{\partial f}{\partial y} = 0 \Rightarrow 4x + 2y = 0 \Rightarrow y = -2x$ ---- (2)

Substitute (2) in (1) $\Rightarrow 3x^2 + 6x - 8x = 0 \Rightarrow x = 0, \frac{2}{3}$

In (2), $x = 0 \Rightarrow y = 0$ and $x = \frac{2}{3} \Rightarrow y = -\frac{4}{3}$

Therefore, Critical points are (0,0), $\left(\frac{2}{3}, -\frac{4}{3}\right)$.

Step:2 Find $rt - s^2$ at each critical point

$r = 6x + 6,\ \ t = 2,\ \ s = 4$

$rt - s^2 = 12x + 12 - 16$

Critical points	$rt - s^2 = 12x - 4$	$r = 6x + 6$	Remark
$(0,0)$	-4, Negative	----	Neither max. nor min.
$\left(\frac{2}{3}, -\frac{4}{3}\right)$	$8 - 4$, Positive.	Positive	Minimum

Step:3 Conclusion

$f(x,y)$ attains minimum at $\left(\frac{2}{3}, -\frac{4}{3}\right)$.

Minimum value $= f\left(\frac{2}{3}, -\frac{4}{3}\right)$

$$= \left(\frac{2}{3}\right)^3 + 3\left(\frac{2}{3}\right)^2 + 4\left(\frac{2}{3}\right)\left(-\frac{4}{3}\right) + \left(-\frac{4}{3}\right)^2$$

$$= -\frac{4}{27}$$

4. **Find the extreme values of the function $f(x,y) = x^4 + y^4 - 2x^2 + 4xy - 2y^2$.**

$f(x,y) = x^4 + y^4 - 2x^2 + 4xy - 2y^2$

$\frac{\partial f}{\partial x} = 4x^3 - 4x + 4y, \ \frac{\partial f}{\partial y} = 4y^3 + 4x - 4y$

$\frac{\partial^2 f}{\partial x^2} = 12x^2 - 4, \ \frac{\partial^2 f}{\partial y^2} = 12y^2 - 4, \ \frac{\partial^2 f}{\partial x \partial y} = 4$

Step:1 **Find critical points**

$p = \frac{\partial f}{\partial x} = 0 \Rightarrow 4x^3 - 4x + 4y = 0 \Rightarrow x^3 - x + y = 0$ ---- (1)

$q = \frac{\partial f}{\partial y} = 0 \Rightarrow 4y^3 + 4x - 4y = 0 \Rightarrow y^3 - y + x = 0$ ---- (2)

$(1) + (2) \Rightarrow y = -x$

In (2), $y = -x \Rightarrow -x^3 + x + x = 0, \ x(2 - x^2) = 0$

Therefore, Critical points are $(0,0)$, $(\sqrt{2}, -\sqrt{2}), (-\sqrt{2}, \sqrt{2})$

Step:2 **Find $rt - s^2$ at each critical point**

$r = 12x^2 - 4, \ t = 12y^2 - 4, \ s = 4$

$rt - s^2 = (12x^2 - 4)(12y^2 - 4) - 16$

Critical points	$rt - s^2 =$ $(12x^2 - 4)(12y^2 - 4) - 16$	$r =$ $12x^2 - 4$	Remark
$(0,0)$	0	----	Doubtful
$(\sqrt{2}, -\sqrt{2})$	$400 - 16$, Positive	Positive	Minimum
$(-\sqrt{2}, \sqrt{2})$	$400 - 16$, Positive	Positive	Minimum

Step:3 **Conclusion**

$f(x,y)$ attains minimum at $(\sqrt{2}, -\sqrt{2})$ and $(-\sqrt{2}, \sqrt{2})$.

Minimum value $= f(\sqrt{2}, -\sqrt{2}) = f(-\sqrt{2}, \sqrt{2}) = 4 + 4 - 4 - 8 - 4 = -8$

5. Find the extreme values of the function $f(x,y) = x^3y^2(1-x-y)$.

$$f(x,y) = x^3y^2 - x^4y^2 - x^3y^3$$

$$\frac{\partial f}{\partial x} = 3x^2y^2 - 4x^3y^2 - 3x^2y^3, \quad \frac{\partial f}{\partial y} = 2x^3y - 2x^4y - 3x^3y^2$$

$$\frac{\partial^2 f}{\partial x^2} = 6xy^2 - 12x^2y^2 - 6xy^3, \quad \frac{\partial^2 f}{\partial y^2} = 2x^3 - 2x^4 - 6x^3y,$$

$$\frac{\partial^2 f}{\partial x \partial y} = 6x^2y - 8x^3y - 9x^2y^2$$

Step:1 Find critical points

$$p = \frac{\partial f}{\partial x} = 0 \Rightarrow 3x^2y^2 - 4x^3y^2 - 3x^2y^3 = 0 \Rightarrow x^2y^2(3-4x-3y) = 0 \quad \text{---- (1)}$$

$$q = \frac{\partial f}{\partial y} = 0 \Rightarrow 2x^3y - 2x^4y - 3x^3y^2 = 0 \Rightarrow x^3y(2-2x-3y) = 0 \quad \text{---- (2)}$$

$3 - 4x - 3y = 0$

0	1
3/4	0

From (1), $x = 0 \text{ or } \frac{3}{4}$, $y = 0 \text{ or } 1$

If $x = 0$, (2) gives $y = 0$ or $2/3$. If $x = 3/4$, (2) gives $y = 0$.

If $y = 0$, (2) gives $x = 0$ or 1. If $y = 2/3$, (2) gives $x = 0$.

Therefore, Critical points are $(0,0), \left(0,\frac{2}{3}\right), \left(\frac{3}{4},0\right), \left(\frac{1}{2},\frac{1}{3}\right), (1,0)$.

Step:2 Find $rt - s^2$ at each critical point

	(0, 0)	(0, 2/3)	(3/4, 0)	(1/2. 1/3)	(1, 0)
$r = 6xy^2(1-2x-y)$	0	0	0	-1/9	0
$t = 2x^3(1-x-3y)$	0	0	1/128	-1/8	0
$s = x^2y(6-8x-9y)$	0	0	0	-1/12	0

Critical points	$rt - s^2$	r	Remark
$(0,0)$	0	0	Saddle point
$(0,2/3)$	0	0	Saddle point
$(3/4,0)$	0	0	Saddle point
$(1/2,1/3)$	Positive	Negative	Maximum
$(1,0)$	0	0	Saddle point

Step:3 Conclusion

$f(x,y)$ attains maximum at $\left(\frac{1}{2},\frac{1}{3}\right)$. Minimum value $= f\left(\frac{1}{2},\frac{1}{3}\right) = \frac{1}{432}$

6. **Find the extreme values of the function $f(x,y) = x^3 + 3xy^2 - 3x^2 - 3y^2 + 4$.**

$$f(x,y) = x^3 + 3xy^2 - 3x^2 - 3y^2 + 4$$

$$\frac{\partial f}{\partial x} = 3x^2 + 3y^2 - 6x, \quad \frac{\partial f}{\partial y} = 6xy - 6y$$

$$\frac{\partial^2 f}{\partial x^2} = 6x - 6, \quad \frac{\partial^2 f}{\partial y^2} = 6x - 6, \quad \frac{\partial^2 f}{\partial x \partial y} = 6y$$

Step:1 Find critical points

$$p = \frac{\partial f}{\partial x} = 0 \Rightarrow 3x^2 + 3y^2 - 6x = 0 \Rightarrow x^2 + y^2 - 2x = 0 \quad \text{---- (1)}$$

$$q = \frac{\partial f}{\partial y} = 0 \Rightarrow 6xy - 6y = 0 \qquad \Rightarrow y = 0, x = 1. \quad \text{---- (2)}$$

Substitute $y = 0$ in (1) $\Rightarrow x = 0$ *or* 2.

Substitute $x = 1$ in (1) $\Rightarrow y = -1$ *or* 1.

Therefore, Critical points are $(0,0), (2,0), (1,-1), (1,1)$.

Step:2 Find $rt - s^2$ at each critical point

$$r = 6x - 6, \; t = 6x - 6, \; s = 6y$$

$$rt - s^2 = 36(x-1)^2 - 36y^2$$

Critical points	$rt - s^2$	$r = 6x - 6$	Remark
$(0,0)$	Positive	-6	Maximum
$(2,0)$	Positive	6	Minimum
$(1,1)$	Negative	-----	
$(1,-1)$	Negative	-----	

Step:3 Conclusion

(i) $f(x,y)$ attains maximum at $(0,0)$. Maximum value = $f(0,0) = 4$

(ii) $f(x,y)$ attains minimum at $(2,0)$. Minimum value = $f(2,0) = 0$

7. **Find the extreme values of $f(x,y) = x^3 + y^3 - 63x - 63y + 12xy$.**

$f(x,y) = x^3 + y^3 - 63x - 63y + 12xy$

$\frac{\partial f}{\partial x} = 3x^2 - 63 + 12y$, $\frac{\partial f}{\partial y} = 3y^2 - 63 + 12x$

$\frac{\partial^2 f}{\partial x^2} = 6x$, $\frac{\partial^2 f}{\partial y^2} = 6y$, $\frac{\partial^2 f}{\partial x \partial y} = 12$

Step:1 Find critical points

$p = \frac{\partial f}{\partial x} = 0 \Rightarrow 3x^2 + 12y = 63 \Rightarrow x^2 + 4y = 21$ ---- (1)

$q = \frac{\partial f}{\partial y} = 0 \Rightarrow 3y^2 + 12x = 63 \Rightarrow y^2 + 4x = 21.$ ---- (2)

$(1) - (2) \Rightarrow (x^2 - y^2) - 4(x - y) = 0 \Rightarrow (x - y)(x + y - 4) = 0$

Therefore, $x = y$ or $y = 4 - x$

If $x = y$, (1) becomes $x^2 + 4x - 21 = 0 \Rightarrow x = 3, -7$

If $y = 4 - x$, (1) becomes $x^2 - 4x - 5 = 0 \Rightarrow x = 5, -1$

Therefore, Critical points are $(3,3), (-7,-7), (5,-1), (-1,5)$.

Step:2 Find $rt - s^2$ at each critical point

$r = 6x$, $t = 6y$, $s = 12$

$rt - s^2 = 36xy - 144$

Critical points	$rt - s^2$	$r = 6x$	Remark
$(3,3)$	Positive	18	Minimum
$(-7,-7)$	Positive	-42	Maximum
$(-1,5)$	Negative	-----	
$(5,-1)$	Negative	-----	

Step:3 Conclusion

(i) $f(x,y)$ attains minimum at $(3,3)$. Minimum value = $f(3,3) = -216$

(ii) $f(x,y)$ attains maximum at $(-7,-7)$. Maximum value = $f(-7,-7) = 784$

Home work:

8. Examine the function $f(x, y) = 1 + \sin(x^2 + y^2)$ for extreme values.
   Ans: $A = (a, b)$ *such that* $a^2 + b^2 = \frac{\pi}{2}$ is the max. point. Max. value is 2.

9. Find the extreme values of the function $f(x, y) = \sin x \sin y \sin(x + y)$.
   Ans: $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$ is the max. point. Max. value is $\frac{3\sqrt{3}}{8}$.

10. Show that $f(x, y) = x^3 + y^3 - 3x - 12y + 20$ has a maximum value at the point $(-1, -2)$ and a minimum value at the point $(1, 2)$. (May 22)