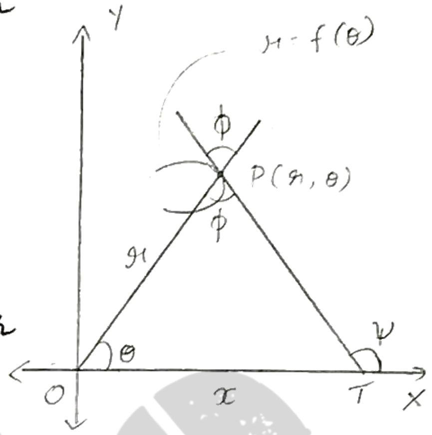
MODULE - 1

1)a) With usual notation prove that $\tan\phi = r\dfrac{d\theta}{dr}$



Let $P(r,\theta)$ be a point on the curve $r=f(\theta)$.
Hence $OP = r$ and $X\hat{O}P = \theta$

Let PT be the tangent to the curve $r = f(\theta)$ and let $X\hat{T}P = \psi$.

Let ϕ be the angle between the radius vector OP and the tangent PT i.e, $O\hat{P}T = \phi$

From the figure,

$$\psi = \phi + \theta$$

$$\tan\psi = \tan(\phi+\theta)$$

$$\tan\psi = \frac{\tan\phi + \tan\theta}{1 - \tan\phi \cdot \tan\theta} \quad \text{——} \quad (1)$$

If $P(x,y)$ are cartesian coordinates of P then we have,

$x = r\cos\theta$, $y = r\sin\theta$

Also, WKT $\tan\psi = \dfrac{dy}{dx}$ = slope of the tangent

Divide both numerator and denominator by $d\theta$

$$\tan\psi = \frac{dy/d\theta}{dx/d\theta}$$

$$\tan\psi = \frac{\frac{d}{d\theta}(r\sin\theta)}{\frac{d}{d\theta}(r\cos\theta)}$$

$$\tan\psi = \frac{r.\cos\theta + \sin\theta\,\frac{dr}{d\theta}}{-r\sin\theta + \cos\theta\,\frac{dr}{d\theta}}$$

Let $\dfrac{dr}{d\theta} = r'$

$$\tan\psi = \frac{r\cos\theta + r'\sin\theta}{r'\cos\theta - r\sin\theta}$$

Divide both numerator and denominator by $r'\cos\theta$

$$\tan\psi = \frac{\frac{r}{r'} + \tan\theta}{1 - \frac{r}{r'}\tan\theta} \quad \text{—— (2)}$$

Comparing (1) and (2),

$$\tan\phi = \frac{r}{r'}$$

$$\Rightarrow \boxed{\tan\phi = r\frac{d\theta}{dr}}$$

1) b) Find the angle between the curves $r = a\log\theta$ and $r = \frac{a}{\log\theta}$

$r = a\log\theta$

take log on both sides

$\log r = \log a + \log(\log\theta)$

diff wrt θ

$$\frac{1}{r}\frac{dr}{d\theta} = 0 + \frac{1}{\log\theta}\left(\frac{1}{\theta}\right)(1)$$

$$\cot\phi_1 = \frac{1}{\theta\log\theta}$$

$$\boxed{\tan\phi_1 = \theta\log\theta}$$

$r = \frac{a}{\log\theta}$

take log on both sides

$\log r = \log a - \log(\log\theta)$

diff wrt θ

$$\frac{1}{r}\frac{dr}{d\theta} = 0 - \frac{1}{\log\theta}\left(\frac{1}{\theta}\right)(1)$$

$$\cot\phi_2 = \frac{-1}{\theta\log\theta}$$

$$\boxed{\tan\phi_2 = -\theta\log\theta}$$

$$\tan|\phi_1 - \phi_2| = \frac{\tan\phi_1 - \tan\phi_2}{1 + \tan\phi_1 \cdot \tan\phi_2}$$

$$= \frac{\theta\log\theta - (-\theta\log\theta)}{1 + \theta\log\theta(-\theta\log\theta)}$$

$$= \frac{2\theta\log\theta}{1 - \theta^2(\log^2\theta)^2}$$

To find θ,

consider $r = a\log\theta$; $r = \frac{a}{\log\theta}$

$$\cancel{a}\log\theta = \frac{\cancel{a}}{\log\theta}$$

$$(\log\theta)^2 = 1$$

$$\Rightarrow \boxed{\theta = e}$$

Substituting in eq^n ① we get,

$$\tan|\phi_1-\phi_2| = \frac{2e\log e}{1-e^2(\log e)^2}$$

but $\log e = 1$

$$\tan|\phi_1-\phi_2| = \frac{2e}{1-e^2} \Rightarrow |\phi_1-\phi_2| = \tan^{-1}\left(\frac{2e}{1-e^2}\right)$$

1)c) Show that radius of curvature at any point θ on the cycloid $x = a(\theta+\sin\theta)$, $y = a(1-\cos\theta)$ is $4a\cos\left(\frac{\theta}{2}\right)$

$$x = a(\theta+\sin\theta) \qquad y = a(1-\cos\theta)$$

$$\frac{dx}{d\theta} = a + a\cos\theta \qquad \frac{dy}{d\theta} = a(\sin\theta)$$

$$y_1 = \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$$

$$y_1 = \frac{\cancel{a}\sin\theta}{\cancel{a}(1+\cos\theta)}$$

$$y_1 = \frac{2\sin\theta/2\cos\theta/2}{2\cos^2\theta/2} \Rightarrow \boxed{y_1 = \tan\theta/2}$$

diff wrt x,

$$y_2 = \sec^2\left(\frac{\theta}{2}\right)\cdot\frac{d\theta}{dx}\cdot\frac{1}{2}$$

$$y_2 = \sec^2\left(\frac{\theta}{2}\right)\cdot\frac{1}{2a(1+\cos\theta)}$$

$$y_2 = \frac{1}{2a}\frac{\sec^2\theta/2}{2\cos^2\theta/2} \Longrightarrow \boxed{y_2 = \frac{1}{4a}\sec^4\left(\frac{\theta}{2}\right)}$$

$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2}$$

$$= \frac{(1+\tan^2\theta/2)^{3/2}}{\frac{1}{4a}\sec^4\left(\frac{\theta}{2}\right)}$$

$$\rho = \frac{4a\left[\sec^2(\theta/2)\right]^{3/2}}{\sec^4\left(\frac{\theta}{2}\right)}$$

$$\rho = \frac{4a}{\sec\theta/2} \Rightarrow \boxed{\rho = 4a\cos\left(\frac{\theta}{2}\right)}$$

2)a) Show that the curves $r = a(1+\sin\theta)$ and $r = a(1-\sin\theta)$ cuts each other orthogonally.

$r = a(1+\sin\theta)$

take log on both sides

$\log r = \log a + \log(1+\sin\theta)$

diff wrt x,

$$\frac{1}{r}\frac{dr}{d\theta} = 0 + \frac{1}{1+\sin\theta}(\cos\theta)$$

$$\cot\phi_1 = \frac{\cos\theta}{1+\sin\theta}$$

$r = a(1-\sin\theta)$

take log on both sides

$\log r = \log a + \log(1-\sin\theta)$

diff wrt x,

$$\frac{1}{r}\frac{dr}{d\theta} = 0 + \frac{1}{1-\sin\theta}(-\cos\theta)$$

$$\cot\phi_2 = \frac{-\cos\theta}{1-\sin\theta}$$

$$\cot\phi_1 \cdot \cot\phi_2 = \frac{\cos\theta}{1+\sin\theta} \times \frac{-\cos\theta}{1-\sin\theta}$$

$$= \frac{-\cos^2\theta}{1^2-\sin^2\theta}$$

$$= \frac{-\cancel{\cos^2\theta}}{\cancel{\cos^2\theta}} \Rightarrow -1 //$$

Since $\cot\phi_1 \cdot \cot\phi_2 = -1$ i.e, $|\phi_1 - \phi_2| = \pi/2$

$\therefore$ The given two curves intersect orthogonally.

2)b) Find the pedal equation of curve $\frac{2a}{r} = (1+\cos\theta)$

$\log 2a - \log r = \log(1+\cos\theta)$

$$0 - \frac{1}{r}\frac{dr}{d\theta} = \frac{1}{1+\cos\theta}(0-\sin\theta)$$

$$\boxed{\cot\phi = \frac{\sin\theta}{1+\cos\theta}}$$

pedal eq^n, $\frac{1}{p^2} = \frac{1}{r^2}[1+\cot^2\phi]$

$$\frac{1}{p^2} = \frac{1}{r^2}\left[1+\frac{\sin^2\theta}{(1+\cos\theta)^2}\right]$$

$$\frac{1}{p^2} = \frac{1}{r^2}\left[\frac{1+\cos^2\theta+2\cos\theta+\sin^2\theta}{(1+\cos\theta)^2}\right]$$

$$\frac{1}{p^2} = \frac{1}{r^2}\left[\frac{2+2\cos\theta}{(1+\cos\theta)^2}\right]$$

$$\frac{1}{p^2} = \frac{2}{r^2}\left[\frac{\cancel{1+\cos\theta}}{(1+\cos\theta)^{\cancel{2}}}\right] \Rightarrow \boxed{\frac{1}{p^2} = \frac{2}{r^2}\left[\frac{1}{1+\cos\theta}\right]}$$

To eliminate θ,

$1+\cos\theta = \frac{2a}{r}$

$$\Rightarrow \frac{1}{p^2} = \frac{\cancel{2}}{r^{\cancel{2}}}\times\frac{\cancel{r}}{\cancel{2}a} \Rightarrow \boxed{p=\sqrt{ar}}$$

2)c) Find the radius of curvature for curve $y^2 = \frac{4a^2(2a-x)}{x}$, where the curve meets x-axis.

$$y^2 = \frac{4a^2(2a-x)}{x}$$

$$xy^2 = 4a^2(2a-x)$$

Since the curve cuts the x axis, we have $y=0$

Substitute $y=0$ in the given curve

$$xy^2 = 4a^2(2a-x)$$

$$x(0)^2 = 4a^2(2a-x)$$

$$4a^2x = 8a^3$$

$$x = \frac{8a^3}{4a^2} \Rightarrow \boxed{x=2a}$$

Therefore, to find the ROC of given curve at $(2a, 0)$

Consider, $xy^2 = 4a^2(2a-x) \Rightarrow xy^2 = 8a^3 - 4a^2x$

Diff wrt to x

$$x \cdot 2yy_1 + y^2 \cdot 1 = 0 - 4a^2$$

$$2xyy_1 + y^2 = -4a^2$$

$$y_1 = \frac{-4a^2 - y^2}{2xy}$$

at $(2a, 0)$ $\quad y_1 = \frac{-4a^2 - (0)^2}{2(2a)(0)}$

$$y_1 = \infty \text{ at } (2a, 0)$$

Since $y_1 = \infty$ at $(2a, 0)$ we consider,

$$x_1 = \frac{dx}{dy} = \frac{2xy}{-4a^2 - y^2}$$

at $(2a, 0)$ $\quad x_1 = \frac{2(2a)(0)}{-4a^2 - 0} \Rightarrow x_1 = 0$

Consider, $x_1 = \frac{2xy}{-4a^2 - y^2} = \frac{2xy}{-(4a^2 + y^2)}$

$$(4a^2 + y^2)x_1 = -2xy$$

Diff wrt y

$$(4a^2 + y^2)x_2 + x_1(2y) = -2[x \cdot 1 + y \cdot x_1]$$

at $(2a, 0)$ $\quad (4a^2 + 0^2)x_2 + 0 = -2[2a \cdot 1 + 0 \cdot x_1]$

$$4a^2 \cdot x_2 = -4a$$

$$x_2 = \cancel{-\frac{4a}{4a^2}} \Rightarrow \boxed{x_2 = -\frac{1}{a}}$$

ROC, $\quad \rho = \frac{(1 + y_1^2)^{3/2}}{y_2}$

$$= \frac{[1+0]^{3/2}}{\left(-\frac{1}{a}\right)} \Rightarrow \boxed{|\rho| = a}$$

3)a) Show that, expand (log (secx) upto term containing x^4 using Maclaurin's series.

→ Maclaurin's series expansion is given by:

$$y(x) = y(0) + xy_1(0) + \frac{x^2}{2!}y_2(0) + \dots \quad \text{--- (1)}$$

$y(x) = \log(\sec x)$; $\quad y(0) = \log(\sec 0) = \boxed{0}$

$y_1(x) = \frac{1}{\cancel{\sec x}} \cdot \cancel{\sec x} \cdot \tan x = \tan x$; $\quad y_1(0) = \tan 0 = \boxed{0}$

$y_2(x) = \sec^2 x$; $\quad y_2(0) = \sec^2(0) = \boxed{1}$

$y_2(x) = 1 + \tan^2 x$

$y_2(x) = 1 + y_1^2$

$y_3(x) = 2y_1y_2$; $\quad y_3(0) = 2(0)(1) = \boxed{0}$

$y_4(x) = 2\{y_1y_3 + y_2^2\}$; $\quad y_4(0) = 2\{0 + 1^2\} = \boxed{2}$

Substitute in eqn (1),

$$y(x) = \frac{x^2}{2!}(1) + \frac{x^2}{4!}(2)$$

$$y(x) = \frac{x^2}{2} + \frac{x^2}{24}(2) \Rightarrow \boxed{y(x) = \frac{x^2}{2} + \frac{x^2}{12}}$$

3)b) If $u = e^{ax+by} f(ax-by)$, prove that $b\frac{\partial u}{\partial x} + a\frac{\partial u}{\partial y} = 2abu$ by using the concept of composite functions.

→ $$u = e^{ax+by} f(ax-by) \quad \text{--- (1)}$$

Diff partially wrt to x

$$\frac{\partial u}{\partial x} = e^{ax+by} f'(ax-by)\frac{\partial}{\partial x}(ax-by) + f(ax-by)\,e^{ax+by}\frac{\partial}{\partial x}(ax+by)$$

$$\frac{\partial u}{\partial x} = e^{ax+by} f'(ax-by)\cdot a + f(ax-by)\,e^{ax+by}\cdot a$$

$$\frac{\partial u}{\partial x} = ae^{ax+by} f'(ax-by) + au$$

Multiply 'b' on both sides

$$b \cdot \frac{\partial u}{\partial x} = abe^{ax+by} f'(ax-by) + abu \quad \text{--- (2)}$$

Diff (1) w.r.t y.

$$\frac{\partial u}{\partial y} = e^{ax+by} f'(ax-by)(-b) + f(ax-by)\, e^{ax+by}(b)$$

$$\frac{\partial u}{\partial y} = -be^{ax+by} f'(ax-by) + bu$$

Multiply 'a' on both sides

$$a \cdot \frac{\partial u}{\partial y} = -abe^{ax+by} f'(ax-by) + abu \quad \text{--- (3)}$$

Add eq^n (2) and (3)

$$\therefore \boxed{b\frac{\partial u}{\partial x} + a\frac{\partial u}{\partial y} = 2abu}$$

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3)c) Find the extreme values of function $f(x,y) = x^3 + 3xy^2 - 3y^2 - 3x^2 + 4$

→ $f(x,y) = x^3 + 3xy^2 - 3y^2 - 3x^2 + 4$

$f_x = 3x^2 + 3y^2 - 6x$

$f_y = 6xy - 6y$

To find stationary points such that $f_x = 0$ and $f_y = 0$

$3x^2 + 3y^2 - 6x = 0$ --- (1) ; $6xy - 6y = 0$

($\div$ by 6)

$xy - y = 0$

$y(x-1) = 0$

$\boxed{y=0}$ $\boxed{x=1}$

Substitute $y=0$ in eq^n (1).

$3x^2 + 3y^2 - 6x = 0$

($\div$ by 3) $x^2 + y^2 - 2x = 0$

$x^2 - 2x = 0$

$x(x-2) = 0 \Rightarrow \boxed{x=0}$ $\boxed{x=2}$

$\therefore$ stationary points are $(0,0)\ (2,0)$

Substitute $x=1$ in eqⁿ ①,

$$3x^2+3y^2-6x=0$$

$(\div 3)$ $$x^2+y^2-2x=0$$

$$1+y^2-2=0$$

$$y^2-1=0 \Rightarrow \boxed{y=\pm 1}$$

$\therefore$ Stationary points are $(1,-1)\ (1,1)$

$\therefore$ stationary points are $(0,0)\ (2,0)\ (1,-1)\ (1,1)$

$A = f_{xx} = 6x-6$

$B = f_{xy} = 6y$

$C = f_{yy} = 6x-6$

	$(0,0)$	$(2,0)$	$(1,-1)$	$(1,1)$
$A=6x-6$	$-6<0$	$6>0$	0	0
$B=6y$	0	0	-6	6
$C=6x-6$	-6	6	0	0
$AC-B^2$	$36>0$	$36>0$	-36	-36
conclusion	Max point	Min point	saddle point	saddle point

$\therefore$ Minimum value of $f(2,0) = x^3+3xy^2-3y^2-3x^2+4$

$= 2^3+2(2)(0)-3(0)-3(2)^2+4$

$= 8+0-0-12+4$

$= 0$

Maximum value of $f(0,0) = 0+3(0)(0)-3(0)-3(0)+4$

$= 4$

4)a) Evaluate

(i) $\lim_{x\to 0}\left(\frac{a^x+b^x}{2}\right)^{\frac{1}{x}}$

$\rightarrow$ Let $K = \lim_{x\to 0} \left(\frac{a^x+b^x}{2}\right)^{\frac{1}{x}}$

$\log_e K = \lim_{x\to 0} \frac{1}{x} \log\left(\frac{a^x+b^x}{2}\right)$

$\log_e K = \lim_{x\to 0} \frac{\log\left(\frac{a^x+b^x}{2}\right)}{x}$

Apply LHR,

$$\log_e K = \lim_{x\to 0} \frac{\frac{1}{\left(\frac{a^x+b^x}{2}\right)} \cdot \frac{1}{2}\{a^x\log a + b^x \log b\}}{1}$$

$\log_e K = \lim_{x\to 0} \frac{1}{(a^x+b^x)} \{a^x \log a + b^x \log b\}$

$\log_e K = \frac{1}{(a^0+b^0)} \{a^0 \log a + b^0 \log b\}$

$\log_e K = \frac{1}{2}\{\log a + \log b\}$

$\log_e K = \frac{1}{2}(\log ab)$

$\log_e K = \log(ab)^{1/2}$

$K = (ab)^{1/2} \implies \boxed{K = \sqrt{ab}}$

(ii) $\lim_{x\to 0} \left(\frac{\tan x}{x}\right)^{\frac{1}{x}}$

Let $K = \lim_{x\to 0} \left(\frac{\tan x}{x}\right)^{\frac{1}{x}}$

$\log_e K = \lim_{x\to 0} \frac{1}{x} \log\left(\frac{\tan x}{x}\right)$

$\log_e K = \lim_{x\to 0} \frac{\log\left(\frac{\tan x}{x}\right)}{x}$

Apply LHR,

$$\log_e K = \frac{\lim\limits_{x\to 0} \frac{1}{\left(\frac{\tan x}{x}\right)} \left\{ \frac{x \cdot \sec^2 x - \tan x \cdot 1}{x^2} \right\}}{1}$$

$$\log_e K = \lim_{x\to 0} \frac{x \cdot \sec^2 x - \tan x}{x^2} \qquad \left(\because \lim_{x\to 0} \frac{\tan x}{x} = 1\right)$$

Apply LHR,

$$\log_e K = \lim_{x\to 0} \frac{x \cdot 2\sec^2 x \cdot \tan x + \sec^2 x \cdot 1 - \sec^2 x}{2x}$$

$$\log_e K = \lim_{x\to 0} \frac{2x \sec^2 x \cdot \tan x}{2x}$$

$$\log_e K = \lim_{x\to 0} \frac{\sec^2 x \cdot \tan x}{1}$$

$$\log_e K = \frac{\sec^2(0) \cdot \tan(0)}{1}$$

$$\log_e K = 0 \implies \boxed{K = e^0 = 1}$$

b) If $u = f(x-y, y-z, z-x)$ show that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$

Let $p = x-y$, $q = y-z$, $r = z-x$

$\therefore \quad u \to (p, q, r) \to (x, y, z) \implies u \to (x, y, z)$

$\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}$ exists

$$\frac{\partial u}{\partial x} = u_x = \frac{\partial u}{\partial p} \cdot \frac{\partial p}{\partial x} + \frac{\partial u}{\partial q} \frac{\partial q}{\partial x} + \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial x}$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial p}(1) + \frac{\partial u}{\partial q}(0) + \frac{\partial u}{\partial r}(-1)$$

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial u}{\partial p} - \frac{\partial u}{\partial r}} \quad \text{——— } (1)$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial p} \cdot \frac{\partial p}{\partial y} + \frac{\partial u}{\partial q} \cdot \frac{\partial q}{\partial y} + \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial y}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial p}(-1) + \frac{\partial u}{\partial q}(1) + \frac{\partial u}{\partial r}(0)$$

$$\boxed{\frac{\partial y}{\partial y} = -\frac{\partial u}{\partial p} + \frac{\partial u}{\partial q}} \quad \text{—— (2)}$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial p}\cdot\frac{\partial p}{\partial z} + \frac{\partial u}{\partial q}\cdot\frac{\partial q}{\partial z} + \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial z}$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial p}(0) + \frac{\partial u}{\partial q}(-1) + \frac{\partial u}{\partial r}(1)$$

$$\boxed{\frac{\partial u}{\partial z} = -\frac{\partial u}{\partial q} + \frac{\partial u}{\partial r}} \quad \text{—— (3)}$$

Add (1), (2) and (3)

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{\partial u}{\partial p} - \frac{\partial u}{\partial r} - \frac{\partial u}{\partial p} + \frac{\partial u}{\partial q} - \frac{\partial u}{\partial q} + \frac{\partial u}{\partial r}$$

$$\therefore \boxed{\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0}$$

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4)c) If $x+y+z=u$, $y+z=v$ and $z=uvw$ find values of $\frac{\partial(x,y,z)}{\partial(u,v,w)}$

$$J\left(\frac{x,y,z}{u,v,w}\right) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

Consider, $x+y+z=u$, $y+z=v$, $z=uvw$

$x+v=u$ $\qquad$ $y=v-z$

$x=v-u$ $\qquad$ $y=v-uvw$

$$\frac{\partial x}{\partial u} = 1-0 = 1 \qquad \frac{\partial y}{\partial u} = 0 - vw = -vw \qquad \frac{\partial z}{\partial u} = vw$$

$$\frac{\partial x}{\partial v} = -1 \qquad \frac{\partial y}{\partial v} = 1 - uw = 1-uw \qquad \frac{\partial z}{\partial v} = uw$$

$$\frac{\partial x}{\partial w} = 0 \qquad \frac{\partial y}{\partial w} = -uv \qquad \frac{\partial z}{\partial w} = uv$$

$$J\left(\frac{u,v,w}{x,y,z}\right) = \begin{vmatrix} 1 & -1 & 0 \\ -vw & 1-uw & -uv \\ vw & uw & uv \end{vmatrix}$$

$$J = 1\left[uv - u^2vw - [-u^2vw]\right] + 1\left[-u\cancel{v}^2w + u\cancel{v}^2w\right] + 0$$

$$J = 1\left[uv - \cancel{u^2}vw + \cancel{u^2}vw\right] + 1[0]$$

$$\boxed{J = uv}$$

5) a) solve $\frac{dy}{dx} + \frac{y}{x} = x^2 y^6$

→ $\frac{dy}{dx} + \frac{y}{x} = x^2 y^6$ ——— ①

given DE is of the form $\frac{dy}{dx} + Py = Qy^n$

eqⁿ ① divide y^6 throughout,

$\frac{1}{y^6}\frac{dy}{dx} + \frac{1}{x} \cdot \frac{1}{y^5} = x^2$ ——— ②

Substitute $\frac{1}{y^5} = t$

diff wrt to x

$\frac{-5}{y^6} \cdot \frac{dy}{dx} = \frac{dt}{dx} \Rightarrow \frac{1}{y^6}\frac{dy}{dx} = -\frac{1}{5} \cdot \frac{dt}{dx}$

Substitute in ①, $-\frac{1}{5}\frac{dt}{dx} + \frac{1}{x} \cdot t = x^2$

(x^{ly} by -5) $\frac{dt}{dx} - 5 \cdot \frac{1}{x} \cdot t = -5x^2$ ——— ③

$\frac{dt}{dx} + Pt = Q$; where $P = -5 \cdot \frac{1}{x}$; $Q = -5x^2$

$IF = e^{\int P \cdot dx} = e^{-5\int \frac{1}{x} dx} = e^{(\log x)^{-5}} = x^{-5} = \frac{1}{x^5}$

solⁿ of ③ is given by -

$t(IF) = \int Q(IF)\, dx + c$

$t \cdot \frac{1}{x^5} = \int -5x^2 \cdot \frac{1}{x^5} \cdot dx + c$

$t \cdot \frac{1}{x^5} = -5 \int \frac{1}{x^3} \cdot dx + c$

$t \cdot \frac{1}{x^5} = -5 \frac{x^{-2}}{-2} \cancel{dx} + c$

$$\boxed{\frac{1}{y^5} \cdot \frac{1}{x^5} = \frac{5}{2} x^{-2} + c}$$

b) Find orthogonal trajectories of $\frac{x^2}{a^2} + \frac{y^2}{b^2+\lambda} = 1$, where λ is a parameter

$$\frac{x^2}{a^2} + \frac{y^2}{b^2+\lambda} = 1 \quad \text{——} \quad (1)$$

$$\frac{x^2(b^2+\lambda) + a^2y^2}{a^2(b^2+\lambda)} = 1$$

$$x^2(b^2+\lambda) + a^2y^2 = a^2(b^2+\lambda)$$

diff wrt x

$$2x\,(b^2+\lambda) + a^2 \cdot 2y\frac{dy}{dx} = 0$$

$(\div 2)$ $$x\,(b^2+\lambda) + a^2 \cdot y\,\frac{dy}{dx} = 0$$

$$(b^2+\lambda)\,x = -a^2y \cdot \frac{dy}{dx}$$

$$(b^2+\lambda) = -\frac{a^2y}{x}\,\frac{dy}{dx}$$

Substitute in equation (1),

$$\frac{x^2}{a^2} + \frac{y^2}{-\frac{a^2y}{x}\cdot\frac{dy}{dx}} = 1$$

$$\frac{x^2}{a^2} - \frac{xy}{a^2}\cdot\frac{dx}{dy} = 1$$

Replace $-\frac{dx}{dy}$ by $\frac{dy}{dx}$

$$\Rightarrow \quad \frac{x^2}{a^2} + \frac{xy}{a^2}\cdot\frac{dy}{dx} = 1$$

$$x^2 + xy\cdot\frac{dy}{dx} = a^2$$

$$xy\cdot\frac{dy}{dx} = a^2 - x^2$$

$$y\cdot dy = \left(\frac{a^2-x^2}{x}\right)dx$$

$$\int y \cdot dy = \int \frac{a^2}{x} \cdot dx - \int x \cdot dx + c$$

$$\frac{y^2}{2} = a^2 \log x - \frac{x^2}{2} + c$$

$$\boxed{y^2 = 2a^2 \log x - x^2 + 2c}$$

c) Solve $xyp^2 - (x^2+y^2)p + xy = 0$

$$xyp^2 + xy = (x^2+y^2)p$$

$$xyp^2 - (x^2+y^2)p + xy = 0$$

It is similar to $ax^2 + bx + c = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Here $a = xy$, $b = -(x^2+y^2)$, $c = xy$

$$p = \frac{(x^2+y^2) \pm \sqrt{(x^2+y^2)^2 - 4(xy)(xy)}}{2xy}$$

$$= \frac{(x^2+y^2) \pm \sqrt{x^2+y^2 + 2x^2y^2 - 4x^2y^2}}{2xy}$$

$$= \frac{(x^2+y^2) \pm \sqrt{(x^2-y^2)^2}}{2xy}$$

$$= \frac{(x^2+y^2) \pm (x^2-y^2)}{2xy}$$

$$p = \frac{(x^2+y^2) + (x^2-y^2)}{2xy}$$

$$p = \frac{2x^2}{2xy} = \frac{x}{y}$$

$$p = \frac{(x^2+y^2) - (x^2-y^2)}{2xy}$$

$$p = \frac{2y^2}{2xy} = \frac{y}{x}$$

$$\frac{dy}{dx} = \frac{x}{y}$$

$$y \cdot dy = x \cdot dx$$

$$\int y \cdot dy = \int x \cdot dx$$

$$\frac{y^2}{2} = \frac{x^2}{2} + c$$

$$\frac{y^2}{\cancel{2}} = \frac{x^2 + 2c}{\cancel{2}}$$

$$y^2 - x^2 - K = 0$$

$$\frac{dy}{dx} = \frac{y}{x}$$

$$\frac{1}{y} \cdot dy = \frac{1}{x} \cdot dx$$

$$\int \frac{1}{y} \cdot dy = \int \frac{1}{x} \cdot dx$$

$$\log y = \log x + c$$

$$\log y = \log x + \log K$$

$$\log y = \log (xK)$$

$$y = xK \Rightarrow y - xK = 0$$

General solution is $(y^2 - x^2 - K)(y - xK) = 0$

---

6)a) solve $(x^2+y^2+x)dx + xy \cdot dy = 0$

$$(x^2+y^2+x)dx + xy \cdot dy = 0 \quad \text{——— (1)}$$

$M = x^2+y^2+x \qquad N = xy$

$$\frac{\partial M}{\partial y} = 2y \qquad \frac{\partial N}{\partial x} = y$$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \Rightarrow$ equation (1) is not exact

$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 2y - y = y$ ——— close to N

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{y}{xy} = \frac{1}{x} = f(x)$$

$\therefore$ IF $= e^{\int f(x) \cdot dx} = e^{\int \frac{1}{x} \cdot dx} = e^{\log x} \Rightarrow \boxed{IF = x}$

Multiply IF to equation (1),

$$(x^3 + xy^2 + x^2)dx + x^2y \cdot dy = 0 \quad \text{——— (2)}$$

$$\frac{\partial M}{\partial y} = 2xy \qquad \frac{\partial N}{\partial x} = 2xy$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \Rightarrow \text{eq}^n \text{ (2) is exact}$$

soln to eqn (1) is -

$$\int_{y\text{-const}} M\,dx + \int (N \text{ terms free from } x)\,dy = C$$

$$\int_{y\text{-const}} (x^3 + xy^2 + x^2)\,dx + \int 0.dy = C$$

$$\Longrightarrow \boxed{\frac{x^4}{4} + \frac{x^2y^2}{2} + \frac{x^3}{3} = C}$$

---

c) Find general solution of equation $(px-y)(py+x) = a^2p$ by reducing into clairaut's form by taking the substitution $X = x^2$, $Y = y^2$?

$$(px-y)(py+x) = a^2p \quad \text{—— (1)}$$

given: $X = x^2$; $Y = y^2$

$$\frac{dX}{dx} = 2x \quad ; \quad \frac{dY}{dy} = 2y$$

WKT, $p = \frac{dy}{dx}$

$$p = \frac{dy}{dY} \cdot \frac{dY}{dX} \cdot \frac{dX}{dx}$$

$$p = \frac{1}{\cancel{2}y} \cdot P \cdot \cancel{2}x$$

$$\boxed{p = \frac{\sqrt{X}}{\sqrt{Y}}\, P}$$

Substitute in (1).

$$\left[\frac{\sqrt{X}.P}{\sqrt{Y}}.\sqrt{X}-\sqrt{Y}\right]\left[\frac{\sqrt{X}.P\cancel{\sqrt{Y}}}{\cancel{\sqrt{Y}}}+\sqrt{X}\right] = a^2\frac{\sqrt{X}}{\sqrt{Y}}.P$$

$$\left[\frac{XP-Y}{\cancel{\sqrt{Y}}}\right]\left[\sqrt{X}P+\sqrt{X}\right] = \frac{a^2\sqrt{X}}{\cancel{\sqrt{Y}}}P$$

$$(PX-Y)\left[\cancel{\sqrt{X}}(P+1)\right] = a^2\cancel{\sqrt{X}}P$$

$$PX-Y = \frac{a^2P}{P+1}$$

$$Y = PX-\frac{a^2P}{P+1}$$

This is in clairaut's form

Replace P by C

General solution is : $Y = CX - \frac{a^2C}{C+1}$

$$\Rightarrow \boxed{y^2 = cx^2 - \frac{a^2c}{c+1}}$$

b) A series circuit with resistance R, inductance L and electromotive force E is governed by D.E

$L\frac{di}{dt} + Ri = E$ where L and R are constants and initially current i is zero. Find the current at time t?

→ $$L\frac{di}{dt} + Ri = E$$

($\div$ by L throughout)

$$\frac{di}{dt} + \frac{Ri}{L} = \frac{E}{L}$$

This is a linear DE in i of form $\frac{dy}{dx} + Py = Q$

Here, $P = \frac{R}{L}$, $Q = \frac{E}{L}$

$$IF = e^{\int P.dt} = e^{\int \frac{R}{L} dt} = e^{\frac{R}{L}.t}$$

Solution: $y(IF) = \int Q(IF)\, dt + c$

$$i(IF) = \int Q(IF)\, dt + c$$

$$i.\, e^{\frac{Rt}{L}} = \int \frac{E}{L} \cdot e^{\frac{Rt}{L}} .\, dt + c$$

$$i.\, e^{\frac{Rt}{L}} = \frac{E}{\cancel{L}} \frac{e^{\frac{Rt}{L}}}{\frac{R}{\cancel{L}}} + c_1$$

$$i.\, e^{\frac{Rt}{L}} = \frac{E}{R} . e^{\frac{Rt}{L}} + c_1$$

($\div$ by $e^{\frac{Rt}{L}}$)

$$i = \frac{E}{R} + c_1 e^{-\frac{Rt}{L}} \quad \text{——— (1)}$$

Eq^n (1) is the general solution of given DE

given:- Initially current is 0

i.e, $i = 0$ when $t = 0$

$$0 = \frac{E}{R} + c_1 e^0 \Rightarrow \boxed{c_1 = \frac{-E}{R}}$$

Substitute c_1 in eq^n (1),

$$i = \frac{E}{R} - \frac{E}{R} . e^{-\frac{Rt}{L}}$$

$$\boxed{i = \frac{E}{R}\left(1 - e^{\frac{-Rt}{L}}\right)}$$

MODULE - 4

7)a) Find the least positive values of x such that

(i) $71 \equiv x \pmod 8$

(ii) $78 + x \equiv 3 \pmod 5$

(iii) $89 \equiv (x+3) \pmod 4$

→ (i) $71 \equiv x \pmod 8$

$\boxed{x = 7}$ is the least positive value

(ii) $78 + x \equiv 3 \pmod 5$

$78 + x - 3 \equiv 0 \pmod 5$

$75 + x \equiv 5 \times K$

By inspection,

$\boxed{x = 5}$ is the least positive value because $75 + 5 = 80 = 5 \times \underline{16}$

(iii) $89 \equiv (x+3) \pmod 4$

$86 - x \equiv 0 \pmod 4$

$86 - x \equiv 4K$

By inspection,

$\boxed{x = 2}$ is the least positive value because $86 - 2 = 84 = 4 \times \underline{21}$

7)b) Find the remainder when $(\overset{349}{394} \times 74 \times 36)$ is divided by 3

→ $349 \equiv 1 \pmod 3$

$74 \equiv 2 \pmod 3$

$36 \equiv 0 \pmod 3$

$\therefore (349 \times 74 \times 36) \equiv (1 \times 2 \times 0) \pmod 3$

$\equiv 0 \pmod 3$

$\therefore$ 0 is the remainder when $349 \times 74 \times 36$ is divided by 3

7)c) Solve $2x + 6y \equiv 1 \pmod 7$

$4x + 3y \equiv 2 \pmod 7$

$\rightarrow$ Here, $a = 2$, $b = 6$, $r = 1$

$c = 4$, $d = 3$, $s = 2$; $n = 7$

$\gcd(a, b, n) = \gcd(2, 6, 7) = 1$ and $1 | 4$

$\therefore$ System has solution.

$\gcd(ad - bc, n) = \gcd(18, 7) = 1$

$\therefore$ The system has unique solution

consider, $2x + 6y \equiv 1 \pmod 7$ $(\times 2)$

$4x + 3y \equiv 2 \pmod 7$

$\cancel{4x} + 12y \equiv 2 \pmod 7$

$\cancel{4x} + 3y \equiv 2 \pmod 7$

$9y \equiv 0 \pmod 7$

By inspection, $\boxed{y = 0}$

$\therefore \boxed{y \equiv 0 \pmod 7}$

consider $2x+6y \equiv 1 \pmod 7$

$2x \equiv 1 \pmod 7$ $\quad (\because y=0)$

By inspection,

$\boxed{x=4}$ satisfies the equation

$\therefore$ $\boxed{x \equiv 4 \pmod 7}$

Thus solution is $x \equiv 4 \pmod 7$; $y \equiv 0 \pmod 7$

---

8)a) (i) Find the last digit of 7^{2013}

(ii) Find the last digit of 13^{37}

→ (i) $7^{2013} = 7^{4\times 503+1}$

$= 7^{4k+1}$

$\equiv 7 \pmod{10}$

$$\begin{array}{r} 4\,)\,2013\,(\,503 \\ 20 \\ \hline 013 \\ 12 \\ \hline 1 \end{array}$$

$\therefore$ 7 is the last digit

(ii) $13 \equiv 13 \pmod{10}$

$13^2 \equiv 13^2 \pmod{10}$

$13^2 \equiv 169 \pmod{10}$

$13^2 \equiv 9 \pmod{10}$

$13^2 \equiv -1 \pmod{10}$

$(13^2)^{18} \equiv (-1)^{18} \pmod{10}$

$13^{36} \equiv 1 \pmod{10}$

$13^{36}.13^1 \equiv 13 \pmod{10}$

$13^{37} \equiv 3 \pmod{10}$

$\therefore$ Unit digit = 3

8)b) Find the remainder when number 2^{1000} is divided by 13

→ $a = 2$, $p = 13$

$\gcd(2, 13) = 1$

$$12 \overline{)1000} (83$$
$$\frac{96}{40}$$
$$\frac{36}{4}$$

By Fermat's little theorem,

$$a^{p-1} \equiv 1 \pmod{p}$$
$$2^{12} \equiv 1 \pmod{13}$$
$$(2^{12})^{83} \equiv 1^{83} \pmod{13}$$
$$2^{996} \equiv 1 \pmod{13}$$
$$2^{996} \cdot 2^4 \equiv 16 \pmod{13}$$
$$2^{1000} \equiv 16 \pmod{13}$$
$$2^{1000} \equiv 3 \pmod{13}$$

$$13 \overline{)16} (1$$
$$\frac{13}{3}$$

∴ 3 is the remainder when 2^{1000} is divided by 13.

8)c) Find the remainder when 14! is divided by 17

→ Here, $p = 17$

By wilson's theorem,

$$(p-1)! \equiv -1 \pmod{p}$$
$$16! \equiv -1 \pmod{17}$$
$$16 \times 15 \times 14! \equiv -1 \pmod{17}$$
$$(-1) \times (-2) \times 14! \equiv -1 \pmod{17}$$
$$2 \times 14! \equiv 16 \pmod{17}$$

$(\div 2) \Rightarrow$ $14! \equiv 8 \pmod{17}$ $\Rightarrow$ 8 is the remainder.

9) a) Find rank of matrix $\begin{bmatrix} 2 & -1 & -3 & -1 \\ 1 & 2 & 3 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix}$

→

$$A = \begin{bmatrix} 2 & -1 & -3 & -1 \\ 1 & 2 & 3 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix}$$

$R_2 \longrightarrow 2R_2 - R_1$; $R_3 \longrightarrow 2R_3 - R_1$

$$A = \begin{bmatrix} 2 & -1 & -3 & -1 \\ 0 & 5 & 9 & -1 \\ 0 & 1 & 5 & 3 \\ 0 & 1 & 1 & -1 \end{bmatrix}$$

$R_3 \longrightarrow 5R_3 - R_2$; $R_4 \longrightarrow 5R_4 - R_2$

$$A \sim \begin{bmatrix} 2 & -1 & -3 & -1 \\ 0 & 5 & 9 & -1 \\ 0 & 0 & 16 & 16 \\ 0 & 0 & -4 & -4 \end{bmatrix}$$

$R_4 \longrightarrow 4R_4 + R_3$

$$A \sim \begin{bmatrix} 2 & -1 & -3 & -1 \\ 0 & 5 & 9 & -1 \\ 0 & 0 & 16 & 16 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_1 \longrightarrow \frac{1}{2}R_1$; $R_2 \longrightarrow \frac{1}{5}R_2$; $R_3 \longrightarrow \frac{1}{16}R_3$

$$A \sim \begin{bmatrix} 1 & -1/2 & -3/2 & -1/2 \\ 0 & 1 & 9/5 & -1/5 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore \boxed{\rho(A) = 3}$

9b) Solve system of equation using gauss-Jordan method

$x + y + z = 10$; $2x - y + 3z = 19$; $x + 2y + 3z = 22$

consider the augmented matrix

$$[A:B] = \left[\begin{array}{ccc:c} 1 & 1 & 1 & 10 \\ 2 & -1 & 3 & 19 \\ 1 & 2 & 3 & 22 \end{array}\right]$$

$R_2 \longrightarrow R_2 - 2R_1$; $R_3 \longrightarrow R_3 - R_1$

$$[A:B] \sim \left[\begin{array}{ccc:c} 1 & 1 & 1 & 10 \\ 0 & -3 & 1 & -1 \\ 0 & 1 & 2 & 12 \end{array}\right]$$

$R_1 \longrightarrow 3R_1 + R_2$; $R_3 \longrightarrow 3R_3 + R_2$

$$[A:B] \sim \left[\begin{array}{ccc:c} 3 & 0 & 4 & 29 \\ 0 & -3 & 1 & -1 \\ 0 & 0 & 7 & 35 \end{array}\right]$$

$R_1 \longrightarrow 7R_1 - 4R_3$; $R_2 \longrightarrow 7R_2 - R_3$

$$[A:B] \sim \left[\begin{array}{ccc:c} 21 & 0 & 0 & 63 \\ 0 & -21 & 0 & -42 \\ 0 & 0 & 7 & 35 \end{array}\right]$$

System of equation are:

$21x = 63$; $-21y = -42$; $7z = 35$

$\Rightarrow \boxed{x=3}$ $\Rightarrow \boxed{y=2}$ $\Rightarrow \boxed{z=5}$

$x = 3$, $y = 2$, $z = 5$ is the solution

9) c) Using power method find largest eigen value and corresponding eigen vector of matrix $A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$

$\rightarrow$

$$AX^{(0)} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 0 \\ 0.5 \end{bmatrix} = \lambda^{(1)} X^{(1)}$$

$$AX^{(1)} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 2.5 \\ 0 \\ 2 \end{bmatrix} = 2.5\begin{bmatrix} 1 \\ 0 \\ 0.8 \end{bmatrix} = \lambda^{(2)} X^{(2)}$$

$$AX^{(2)} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0.8 \end{bmatrix} = \begin{bmatrix} 2.8 \\ 0 \\ 2.6 \end{bmatrix} = 2.8\begin{bmatrix} 1 \\ 0 \\ 0.929 \end{bmatrix} = \lambda^{(3)} X^{(3)}$$

$$Ax^{(3)} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0.929 \end{bmatrix} = \begin{bmatrix} 2.929 \\ 0 \\ 2.858 \end{bmatrix} = 2.929\begin{bmatrix} 1 \\ 0 \\ 0.976 \end{bmatrix} = \lambda^{(4)}x^{(4)}$$

$$Ax^{(4)} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0.976 \end{bmatrix} = \begin{bmatrix} 2.976 \\ 0 \\ 2.952 \end{bmatrix} = 2.976\begin{bmatrix} 1 \\ 0 \\ 0.992 \end{bmatrix} = \lambda^{(5)}x^{(5)}$$

$$Ax^{(5)} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0.992 \end{bmatrix} = \begin{bmatrix} 2.992 \\ 0 \\ 2.984 \end{bmatrix} = 2.992\begin{bmatrix} 1 \\ 0 \\ 0.997 \end{bmatrix} = \lambda^{(6)}x^{(6)}$$

After 6 iterations, the appropriate eigen value is $\lambda = 2.992$ and the corresponding eigen vector is $x = \begin{bmatrix} 1 \\ 0 \\ 0.997 \end{bmatrix}$

10)a) Solve the following system of equations by gauss seidel method
$10x + y + z = 12$, $x + 10y + z = 12$, $x + y + 10z = 12$

→ The given system of equation are diagonally dominant. Gauss - siedel method is given by -

$$x = \frac{12-y-z}{10} \quad —(1) \qquad y = \frac{12-x-z}{10} \quad —(2) \qquad z = \frac{12-x-y}{10} \quad —(3)$$

Let the initial approxiamation be $((x^{(0)}, y^{(0)}, z^{(0)}) = (0,0,0))$

Ist iteration: $x^{(1)} = \frac{12-0-0}{10} = 1.2$

$$y^{(1)} = \frac{12-1.2-0}{10} = 1.08$$

$$z^{(1)} = \frac{12-1.2-1.08}{10} = 0.972$$

$\therefore (x^{(1)}, y^{(1)}, z^{(1)}) = (1.2, 1.08, 0.972)$

IInd iteration: $x^{(2)} = \frac{12-1.08-0.972}{10} = 0.9948$

$$y^{(2)} = \frac{12 - 0.9948 - 0.972}{10} = 1.0033$$

$$z^{(2)} = \frac{12 - 0.9948 - 1.0033}{10} = 1.0001$$

$$\therefore (x^{(2)}, y^{(2)}, z^{(2)}) = (0.9948, 1.0033, 1.0001)$$

III rd iteration : $x^{(3)} = \frac{12 - 1.0033 - 1.0001}{10} = 0.9996$

$$y^{(3)} = \frac{12 - 0.9996 - 1.0001}{10} = 1.0000$$

$$z^{(3)} = \frac{12 - 0.9996 - 1.0000}{10} = 1.0000$$

$$\therefore (x^{(3)}, y^{(3)}, z^{(3)}) = (0.9996, 1.0000, 1.0000)$$

10) b) For what values of a and b the system of equation :

$x+y+z=6$; $x+2y+3z=10$; $x+2y+az=b$ has -

(i) no solution (ii) a unique solution (iii) infinite number of solution

→ consider the augmented matrix

$$[A:B] = \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 1 & 2 & 3 & : & 10 \\ 1 & 2 & a & : & b \end{bmatrix}$$

$$R_2 \longrightarrow R_2 - R_1 \; ; \; R_3 \longrightarrow R_3 - R_1$$

$$[A:B] = \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 0 & 1 & 2 & : & 4 \\ 0 & 1 & a-1 & : & b-6 \end{bmatrix}$$

$$R_3 \longrightarrow R_3 - R_2$$

$$[A:B] = \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 0 & 1 & 2 & : & 4 \\ 0 & 0 & a-3 & : & b-10 \end{bmatrix}$$

(i) NO solution:

we must have $\rho(A) \neq \rho[A:B]$

If $a=3$, then $\rho(A)=2$

If $b \neq 10$, then $\rho[A:B]=3$

$\therefore$ System of equations have no solution if $a=3$, $b \neq 10$

(ii) Unique solution:

we must have $r=n=3$ i.e $\rho(A)=\rho[A:B]=r=n=3$

If $a \neq 3$, then $\rho(A)=3$

Irrespective of the value of μ, $\rho[A:B]=3$

$\therefore$ system has unique solution if $a \neq 3$ and for any b

(iii) infinite solution:

we must have $r<n$

Since $n=3$, we must have $r=2$

$\rho(A)=2$ if $a=3$

$\rho[A:B]=2$ if $b=10$

$\therefore$ Thus system will have infinite solution if $a=3$ and $b=10$.

10)c) Solve the system of equations by gauss elimination method

$x+y+z=9$, $x-2y+3z=8$, $2x+y-z=3$

→ consider the augmented matrix

$$[A:B] = \begin{bmatrix} 1 & 1 & 1 & : & 9 \\ 1 & -2 & 3 & : & 8 \\ 2 & 1 & -1 & : & 3 \end{bmatrix}$$

$R_2 \rightarrow R_2 - R_1$; $R_3 \rightarrow R_3 - 2R_1$

$$[A:B] = \begin{bmatrix} 1 & 1 & 1 & : & 9 \\ 0 & -3 & 2 & : & -1 \\ 0 & -1 & -3 & : & -15 \end{bmatrix}$$

$R_3 \longrightarrow 3R_3 - R_2$

$$[A:B] \sim \left[\begin{array}{ccc:c} 1 & 1 & 1 & 9 \\ 0 & -3 & 2 & -1 \\ 0 & 0 & -11 & -44 \end{array}\right]$$

This is in upper triangular matrix

$$x+y+z=9$$
$$-3y+2z=-1$$
$$-11z=-44$$

$\Rightarrow \boxed{z=4}$

$\Rightarrow -3y+2(4)=-1$

$-3y=-9$

$\boxed{y=3}$

$\Rightarrow x+y+z=9$

$x+3+4=9$

$\boxed{x=2}$

$\therefore$ Solution is $x=2$, $y=3$, $z=4$.