BMATS-201 Mathematics - II for CSE Stream

Introduction to Integral calculus in CSE

Multiple Integrals :-

Evaluation of double and triple integrals, Evaluation of double integrals by change of order of integration, changing into polar co-ordinates. Applications to find Area and volume by double integral. problems.

Beta and Gamma functions :-

Definitions, properties, relation between Beta and Gamma functions, problems.

Multiple Integrals :-

A repeated process of integration of a function of 2 and 3 variables refer to as double and triple integrals respectively.

1. Evaluate $\int_0^1 \int_x^{\sqrt{x}} xy \, dy \, dx$

→ $I = \int_{x=0}^{1} \int_{y=x}^{\sqrt{x}} x y \, dy \, dx$

$$= \int_0^1 x \left. \frac{y^2}{2} \right|_x^{\sqrt{x}} dx$$

$$= \frac{1}{2} \int_0^1 x \left[(\sqrt{x})^2 - x^2 \right] dx$$

$$= \frac{1}{2} \int_0^1 x^2 - x^3 \, dx$$

$$= \frac{1}{2} \left[\frac{x^3}{3} - \frac{x^4}{4} \right]_0^1$$

$$= \frac{1}{2}\left[\frac{1}{3} - \frac{1}{4}\right] = \frac{1}{2}\left[\frac{4-3}{12}\right] = \frac{1}{24}.$$

2. Evaluate $\int_0^1\int_x^{\sqrt{x}} (x^2+y^2)\,dx\,dy$

→ $I=\int_{x=0}^{1}\int_{y=x}^{\sqrt{x}}(x^2+y^2)\,dy\,dx$

$$I=\int_0^1 x^2y+\frac{y^3}{3}\Big|_x^{\sqrt{x}}\,dx$$

$$I=\int_0^1 x^2\sqrt{x}+\frac{(\sqrt{x})^3}{3}-\left(x^2(x)+\frac{x^3}{3}\right)dx$$

$$=\int_0^1 x^2(x)^{1/2}+\frac{(x)^{3/2}}{3}-x^3-\frac{x^3}{3}\,dx$$

$$I=\int_0^1 x^{5/2}+\frac{x^{3/2}}{3}-\frac{4x^3}{3}\,dx$$

$$I=\frac{x^{7/2}}{7/2}+\frac{1}{3}\frac{x^{5/2}}{5/2}-\frac{\cancel{4}}{3}\frac{x^4}{\cancel{4}}\Big|_0^1$$

$$I=\frac{2}{7}(1)^{7/2}+\frac{1}{3}\frac{2}{5}(1)^{5/2}-\frac{1}{3}(1)^4$$

$$I=\frac{2}{7}+\frac{2}{15}-\frac{1}{3}=\frac{30+14-35}{105}=\frac{\cancel{9}\,3}{\cancel{105}\,35}=\frac{3}{35}$$

$$\boxed{I=\frac{3}{35}}$$

3. Evaluate $\int_0^1\int_0^{\sqrt{1-y^2}} x^3y\,dy\,dx$

$$I=\int_{y=0}^{1}\int_{x=0}^{\sqrt{1-y^2}} x^3y\,dx\,dy$$

$$I=\int_0^1 \frac{x^4}{4}y\Big|_0^{\sqrt{1-y^2}}dy$$

$$I=\frac{1}{4}\int_0^1 \left(\sqrt{1-y^2}\right)^4 y\,dy$$

$(1-y^2)^2=1+y^4-2y^2$

$$I=\frac{1}{4}\int_0^1 y\,(1-y^2)^2\,dy$$

$$I=\frac{1}{4}\int_0^1 y\,(1+y^4-2y^2)\,dy$$

$$I=\frac{1}{4}\int_0^1 y+y^5-2y^3\,dy$$

$$I = \frac{1}{4}\left[\frac{y^2}{2} + \frac{y^6}{6} - \frac{\cancel{2}y^4}{\cancel{4}2}\right]_0^1$$

$$I = \frac{1}{4}\left[\frac{1}{\cancel{2}} + \frac{1}{6} - \frac{\cancel{1}}{2}\right] = \frac{1}{4}\left[\frac{1}{6}\right] = \frac{1}{24}$$

$$\boxed{I = \frac{1}{24}}$$

**** 4. Evaluate $\int_{-1}^{1}\int_{0}^{z}\int_{x-z}^{x+z}(x+y+z)\,dy\,dz\,dx$ [Dec16, 18, June18, Jan21]

$$I = \int_{-1}^{1}\int_{0}^{z}(x+z)y + \frac{y^2}{2}\Big|_{x-z}^{x+z} dz\,dx$$

$$I = \int_{-1}^{1}\int_{0}^{z}\left[(x+z)(\cancel{x}+z-\cancel{x}+z) + \frac{(x+z)^2-(x-z)^2}{2}\right]dz\,dx$$

$$I = \int_{-1}^{1}\int_{0}^{z} 2z(x+z) + \frac{\cancel{4}^{2}xz}{\cancel{2}}\,dx\,dz$$

$(x+z)^2 - (x-z)^2$
$= \cancel{x^2}+\cancel{z^2}+2xz - (\cancel{x^2}+\cancel{z^2}-2xz)$
$= 4xz$

$$I = \int_{-1}^{1} \cancel{2}z\frac{(x+z)^2}{\cancel{2}} + \cancel{2}z\frac{x^2}{\cancel{2}}\Big|_0^z dz$$

$$I = \int_{-1}^{1} z(z+z)^2 + z(z^2)\,dz - (z(z)^2+0)\,dz$$

$$I = \int_{-1}^{1} 4z^3 + \cancel{z^3} - \cancel{z^3}\,dz$$

$$I = \cancel{4}\frac{z^4}{\cancel{4}}\Big|_{-1}^{1}$$

$$I = [1-1] = 0$$

$$\boxed{I = 0}$$

** June19, sep21
5. Evaluate $\int_{-c}^{c}\int_{-b}^{b}\int_{-a}^{a}(x^2+y^2+z^2)\,dy\,dz\,dx$

$\rightarrow$ $$I = \int_{x=-c}^{c}\int_{y=-b}^{b}\int_{z=-a}^{a}(x^2+y^2+z^2)\,dz\,dy\,dx$$ (Right to left)

$$= \int_{x=-c}^{c}\int_{y=-b}^{b} x^2z + y^2z + \frac{z^3}{3}\Big|_{-a}^{a} dy\,dx$$

$$= \int_{-c}^{c}\int_{-b}^{b} \left(x^2a+y^2a+\frac{a^3}{3}\right)-\left(-ax^2-ay^2-\frac{a^3}{3}\right) dy\, dx$$

$$= \int_{-c}^{c}\int_{-b}^{b} \left(x^2a+y^2a+\frac{a^3}{3}+ax^2+ay^2+\frac{a^3}{3}\right) dy\, dx$$

$$= \int_{-c}^{c}\int_{-b}^{b} \left(2ax^2+2ay^2+\frac{2a^3}{3}\right) dy\, dx$$

$$= \int_{-c}^{c} 2ax^2y+\frac{2ay^3}{3}+\frac{2a^3y}{3}\Big|_{-b}^{b} dx$$

$$= \int_{-c}^{c} \left(2ax^2b+\frac{2ab^3}{3}+\frac{2a^3b}{3}\right)-\left(-2ax^2b-\frac{2ab^3}{3}-\frac{2}{3}a^3b\right) dx$$

$$= \int_{-c}^{c} \left(2ax^2b+\frac{2ab^3}{3}+\frac{2a^3b}{3}+2ax^2b+\frac{2ab^3}{3}+\frac{2a^3b}{3}\right) dx$$

$$= \int_{-c}^{c} \left(4ax^2b+\frac{4ab^3}{3}+\frac{4a^3b}{3}\right) dx$$

$$= \frac{4abx^3}{3}+\frac{4ab^3x}{3}+\frac{4a^3bx}{3}\Big|_{-c}^{c}$$

$$= \left(\frac{4abc^3}{3}+\frac{4ab^3c}{3}+\frac{4a^3bc}{3}\right)-\left(-\frac{4abc^3}{3}-\frac{4ab^3c}{3}-\frac{4a^3bc}{3}\right)$$

$$= \left(\frac{8abc^3}{3}+\frac{8ab^3c}{3}+\frac{8a^3bc}{3}\right)$$

$$= \frac{8abc}{3}\left[a^2+b^2+c^2\right]$$

6. Evaluate $\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}} xyz\, dx\, dy\, dz$ ** [June 18, Dec 19]

$$I = \int_0^1\int_0^{\sqrt{1-x^2}} xy\frac{z^2}{2}\Big|_0^{\sqrt{1-x^2-y^2}} dy\, dx$$

$$I = \frac{1}{2}\int_0^1\int_0^{\sqrt{1-x^2}} xy(1-x^2-y^2)-0\; dy\, dx$$

$$I = \frac{1}{2}\int_0^1\int_0^{\sqrt{1-x^2}} xy-x^3y-xy^3\; dy\, dx$$

$$= \frac{1}{2}\int_0^1 \left. \frac{xy^2}{2} - \frac{x^3y^2}{2} - \frac{xy^4}{4} \right|_0^{\sqrt{1-x^2}} dx$$

$$= \frac{1}{2}\int_0^1 \frac{x}{2}(1-x^2) - \frac{x^3}{2}(1-x^2) - \frac{x}{4}(1-x^2)^2 \, dx$$

$1+x^4-2x^2$

$$= \frac{1}{2}\int_0^1 \frac{x}{2} \cancel{-\frac{x^3}{2}} \cancel{-\frac{x^3}{2}} + \frac{x^5}{2} - \frac{x}{4} - \frac{x^5}{4} + \cancel{\frac{2x^3}{4}} \, dx$$

$$= \frac{1}{2}\int_0^1 \left(-\frac{x^3}{2} + \frac{x^5}{4} + \frac{x}{4}\right) dx$$

$$= \frac{1}{2}\left[-\frac{x^4}{8} + \frac{x^6}{24} + \frac{x^2}{8}\right]_0^1$$

$$= \frac{1}{2}\left[\cancel{-\frac{1}{8}} + \frac{1}{24} + \cancel{\frac{1}{8}}\right] = \underline{\underline{\frac{1}{48}}}$$

7. Evaluate $\int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} xyz \, dz \, dy \, dx$ * Dec 17

$$I = \int_0^a \int_0^{\sqrt{a^2-x^2}} xy \frac{z^2}{2} \Big|_0^{\sqrt{a^2-x^2-y^2}} dy \, dx$$

$$I = \frac{1}{2} \int_0^a \int_0^{\sqrt{a^2-x^2}} xy(a^2-x^2-y^2) \, dy \, dx$$

$a^2-x^2-y^2 = t$

$-2y\,dy = dt$

$y\,dy = \frac{dt}{-2}$

$$I = \frac{1}{2} \int_0^a \int_0^{\sqrt{a^2-x^2}} x \frac{t}{-2} dt \, dx$$

$$I = \frac{1}{4} \int_0^a x \frac{t^2}{2(-1)} \Big|_0^{\sqrt{a^2-x^2}} dx$$

$$I = \frac{1}{8} \int_0^a x \frac{(a^2-x^2-y^2)^2}{(-1)} \Big|_0^{\sqrt{a^2-x^2}} dx$$

$$I = \frac{1}{8} \int_0^a \frac{x}{(-1)} \left[(0) - (a^2-x^2)^2\right] dx$$

$$I = \frac{1}{8} \int_0^a x(a^2-x^2)^2 \, dx$$

put $a^2-x^2 = t$

$-2x\,dx = dt$

$x\,dx = \frac{dt}{-2}$

$$I = \frac{1}{8} \int_0^a \frac{t^2 \, dt}{-2}$$

$$I = \frac{1}{16} \frac{t^3}{(-1)3} \Big|_0^a$$

$$I = \frac{1}{16} \left[\frac{(a^2-x^2)^3}{(-1)3}\right]_0^a = \frac{1}{48}\left[0 + (a^2)^3\right] = \frac{a^6}{48}$$

$$\boxed{I = \frac{a^6}{48}}$$

8. Evaluate $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \frac{dz \, dy \, dx}{\sqrt{(1-x^2-y^2)-z^2}}$ * [June 18]

→

$$I = \int_{x=0}^1 \int_{y=0}^{\sqrt{1-x^2}} \sin^{-1}\left(\frac{z}{\sqrt{1-x^2-y^2}}\right) \Big|_0^{\sqrt{1-x^2-y^2}} dy \, dx$$

$$I = \int_0^1 \int_0^{\sqrt{1-x^2}} \left[\sin^{-1}(1) - \cancel{\sin^{-1}(0)}^{\,0}\right] dy\, dx$$

$$\sin^{-1}(1) = \sin^{-1}(\sin \pi/2) = \pi/2$$

$$I = \int_0^1 \int_0^{\sqrt{1-x^2}} \frac{\pi}{2}\, dy\, dx$$

$$I = \int_0^1 \frac{\pi}{2}\, y \Big|_0^{\sqrt{1-x^2}}$$

$$I = \frac{\pi}{2} \int_0^1 \sqrt{1-x^2}\, dx$$

$$\int \sqrt{1-x^2} = \frac{1}{2}\left[x\sqrt{1-x^2} + \sin^{-1}\left(\frac{x}{1}\right)\right]$$

$$I = \frac{\pi}{2}\left[\cancel{\frac{x}{2}\sqrt{1-x^2}}^{\,0} + \frac{1}{2}\sin^{-1}\left(\frac{x}{1}\right)\right]_0^1$$

$$I = \frac{\pi}{2}\left[0 + \frac{1}{2}\cancel{\sin^{-1}(1)}^{\,\pi/2} - \left(0 - \frac{1}{2}\sin^{-1}(0)\right)\right]$$

$$I = \frac{\pi}{2}\,\frac{1}{2}\left[\cancel{\sin^{-1}(1)}^{\,\pi/2} + \cancel{\sin^{-1}(0)}^{\,0}\right]$$

$$I = \frac{\pi}{4}\;\frac{\pi}{2}$$

$$\boxed{I = \frac{\pi^2}{8}}$$

9. Evaluate $\displaystyle\int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} \frac{dz\, dy\, dx}{\sqrt{(a^2-x^2-y^2)-z^2}}$ * [June 17]

$$I = \int_{x=0}^{a} \int_{y=0}^{\sqrt{a^2-x^2}} \int_{z=0}^{\sqrt{a^2-x^2-y^2}} \frac{1}{\sqrt{(a^2-x^2-y^2)-z^2}}\, dz\, dy\, dx$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\, dx = \sin^{-1}(x/a)$$

$$I = \int_0^a \int_0^{\sqrt{a^2-x^2}} \sin^{-1} \frac{z}{\sqrt{(a^2-x^2-y^2)}} \Bigg|_0^{\sqrt{a^2-x^2-y^2}} dy\, dx$$

$$I = \int_0^a \int_0^{\sqrt{a^2-x^2}} \left[\sin^{-1}(1)^{\,\pi/2} - \cancel{\sin^{-1}(0)}^{\,0}\right] dy\, dx$$

$$I = \frac{\pi}{2} \int_0^a y \Big|_0^{\sqrt{a^2-x^2}} dx$$

$$\int \sqrt{a^2-x^2}\, dx = \frac{1}{2}\left(x\sqrt{a^2-x^2} + \sin^{-1}\left(\frac{x}{a}\right)\right)$$

$$I = \frac{\pi}{2} \int_0^a \sqrt{a^2-x^2}\, dx$$

$$I = \frac{\pi}{2}\left[\frac{1}{2}\left(x\sqrt{a^2-x^2} + \sin^{-1}\left(\frac{x}{a}\right)\right)\right]_0^a$$

$$= \frac{\pi}{4}\left[a\cancel{\sqrt{a^2-a^2}}^{\,0} + \sin^{-1}\left(\frac{a}{a}\right)^{\pi/2} - \left(0+\sin^{-1}(0)^{\,0}\right)\right]$$

$$I = \frac{\pi}{4}\left(\frac{\pi}{2}\right)$$

$$\boxed{I = \frac{\pi^2}{8}}$$

10. Evaluate $\int_0^a\int_0^x\int_0^{x+y} e^{x+y+z}\,dz\,dy\,dx$

$\rightarrow$ $$I = \int_0^a\int_0^x\int_0^{x+y} e^{x+y+z}\,dz\,dy\,dx$$

$$I = \int_0^a\int_0^x \left[e^{x+y}\, e^{z}\Big|_0^{x+y}\right] dy\,dx$$

$$I = \int_0^a\int_0^x e^{x+y}\left[e^{x+y} - e^{0}\right] dy\,dx \qquad e^0 = 1$$

$$I = \int_0^a\int_0^x \left[e^{2x+2y} - e^{x+y}\right] dy\,dx$$

$$I = \int_0^a \left[e^{2x}\left(\frac{e^{2y}}{2}\right) - e^{x}\cdot e^{y}\right]_0^x dx$$

$$I = \int_0^a \left[\frac{e^{2x}}{2}\left(e^{2x} - e^{0}\right) - e^{x}\left(e^{x} - e^{0}\right)\right]$$

$$I = \int_0^a \left[\frac{e^{4x}}{2} - \frac{e^{2x}}{2} - e^{2x} + e^{x}\right]$$

$$I = \frac{e^{4x}}{8} - \frac{e^{2x}}{4} - \frac{e^{2x}}{2} + e^{x}\Big|_0^a$$

$$I = \frac{e^{4a}}{8} - \frac{e^{2a}}{4} - \frac{e^{2a}}{2} + e^{a} - \left(\frac{e^0}{8} - \frac{e^0}{4} - \frac{e^0}{2} + 1\right)$$

$$I = \frac{e^{4a}}{8} - \frac{3e^{2a}}{4} + e^{a} - \left(\frac{1}{8} - \frac{1}{4} - \frac{1}{2} + 1\right)$$

$$\boxed{I = \frac{e^{4a}}{8} - \frac{3e^{2a}}{4} + e^{a} - 3/8}$$ OR

$$\boxed{I = \frac{1}{8}\left[e^{4a} - 6e^{2a} + 8e^{a} - 3\right]}$$

11. Evaluate $\int_0^4 \int_0^{2\sqrt{z}} \int_0^{\sqrt{4z-x^2}} dy\, dx\, dz$

→ $I = \int_{z=0}^{4} \int_{x=0}^{2\sqrt{z}} \int_{y=0}^{\sqrt{4z-x^2}} dy\, dx\, dz$

$$= \int_0^4 \int_0^{2\sqrt{z}} y \Big|_0^{\sqrt{4z-x^2}} dx\, dz$$

$$= \int_0^4 \int_0^{2\sqrt{z}} \left[\sqrt{4z-x^2} - 0\right] dx\, dz$$

Let $4z = a^2 \Rightarrow a = 2\sqrt{z}$

$$I = \int_0^4 \int_0^a \left(\sqrt{a^2-x^2}\right) dx\, dz$$

$$I = \int_0^4 \left[\frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2} \sin^{-1}\frac{x}{a}\right]_0^a dz$$

$$I = \int_0^4 \left(0 + \frac{a^2}{2}\left(\sin^{-1}(1) - \sin^{-1}(0)\right)\right) dz$$

$$I = \int_0^4 \frac{a^2}{2}\left(\frac{\pi}{2}\right) dz$$

$$I = \int_0^4 \frac{\pi a^2}{4} dz$$

$$I = \int_0^4 \frac{\pi \cancel{4} z}{\cancel{4}} dz$$

$$I = \pi \frac{z^2}{2}\Big|_0^4$$

$$I = \frac{\pi}{2}(4^2 - 0)$$

$$I = \frac{\pi \cancel{16}^{8}}{\cancel{2}}$$

$$\boxed{I = 8\pi}$$

12. Evaluate $\int_0^1\int_0^{1-x}\int_0^{1-x-y} \frac{dz\,dy\,dx}{(1+x+y+z)^3}$

$\rightarrow$ $I = \int_{x=0}^{1}\int_{y=0}^{1-x}\int_{z=0}^{1-x-y} \frac{1}{(1+x+y+z)^3}\,dz\,dy\,dx$

$I = \int_0^1\int_0^{1-x} \left[\frac{-1}{2(1+x+y+z)^2}\right]_0^{1-x-y} dy\,dx$

$I = \int_0^1\int_0^{1-x} -\frac{1}{8} + \frac{1}{2(1+x+y)^2}\,dy\,dx$

$\frac{-1}{2(1+\cancel{x}+\cancel{y}+1-\cancel{x}-\cancel{y})^2} - \left(\frac{-1}{2(1+x+y)^2}\right)$

$= -\frac{1}{8} + \frac{1}{2(1+x+y)^2}$

$I = \int_0^1 \left[-\frac{1}{8}y - \frac{1}{2(1+x+y)}\right]_0^{1-x} dx$

$-\frac{1}{8}(1-x) - \frac{1}{2(1+\cancel{x}+1-\cancel{x})} - \left(0 - \frac{1}{2(1+x)}\right)$

$= -\frac{1}{8} + \frac{x}{8} - \frac{1}{4} + \frac{1}{2(1+x)}$

$= -\frac{3}{8} + \frac{x}{8} + \frac{1}{2(1+x)}$

$I = \int_0^1 \left(-\frac{3}{8} + \frac{x}{8} + \frac{1}{2(1+x)}\right) dx$

$I = \left. -\frac{3x}{8} + \frac{x^2}{16} + \frac{1}{2}\log(1+x)\right|_0^1$

$I = \left[-\frac{3}{8} + \frac{1}{16} + \frac{1}{2}\left[\log(1+1) - \log(1)^{0}\right]\right]$

$I = -\frac{3}{8} + \frac{1}{16} + \frac{1}{2}\log 2$

$\frac{1}{2}\log 2 = \log 2^{1/2} = \log\sqrt{2}$

$\boxed{I = \left(-\frac{5}{16}\right) + \log\sqrt{2}}$ (OR)

$\boxed{I = \log\sqrt{2} - (5/16)}$

13. Evaluate $\int_0^{\pi/2}\int_0^{a\sin\theta}\int_0^{\frac{a^2-r^2}{a}} r\,dr\,d\theta\,dz$

$\rightarrow$ $I = \int_{\theta=0}^{\pi/2}\int_{r=0}^{a\sin\theta}\int_{z=0}^{\frac{a^2-r^2}{a}} r\,dz\,dr\,d\theta$

$$I = \int_0^{\pi/2}\int_0^{a\sin\theta} rz\Big]_0^{\frac{a^2-r^2}{a}} dr\,d\theta$$

$$I = \int_0^{\pi/2}\int_0^{a\sin\theta} r\left(\frac{a^2-r^2}{a}\right) dr\,d\theta$$

$$I = \frac{1}{a}\int_0^{\pi/2}\int_0^{a\sin\theta} ra^2 - r^3\,dr\,d\theta$$

$$I = \frac{1}{a}\int_0^{\pi/2} \frac{r^2a^2}{2} - \frac{r^4}{4}\Big]_0^{a\sin\theta} d\theta$$

$$I = \frac{1}{a}\int_0^{\pi/2} \frac{a^4\sin^2\theta}{2} - \frac{a^4\sin^2\theta}{4}\,d\theta$$

$$\left[\int_0^{\pi/2}\sin^n\theta\,d\theta = \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdots k \quad \begin{matrix} k=\pi/2 & n=\text{even} \\ k=1 & n=\text{odd}\end{matrix}\right] \text{ standard formula}$$

Reduction formula for $\sin\theta$ and $\cos\theta$

$$I = \frac{1}{a}\left[\frac{a^4}{2}\,\frac{1}{2}\,\frac{\pi}{2} - \frac{a^4}{4}\,\frac{3}{4}\,\frac{1}{2}\,\frac{\pi}{2}\right]$$

$$I = \frac{1}{a}\left[\frac{\pi a^4}{8} - \frac{3a^4\pi}{64}\right]$$

$$I = \frac{1}{a}\left[\frac{8\pi a^4 - 3a^4\pi}{64}\right] = \frac{1}{\cancel{a}}\left[\frac{5\pi a^4}{64}\right]$$

$$\boxed{I = \frac{5a^3\pi}{64}}$$

Evaluation of double integral by change of order of integration

Step1:- Given the integral in either of the forms as

$$I=\iint_R f(x,y)\,dx\,dy=\int_{x=a}^{b}\int_{y=y_1(x)}^{y_2(x)} f(x,y)\,dy\,dx \longrightarrow (1)$$

we have to identify the region of integration R by writing the figure and express (1) in the form

$$I=\iint_R f(x,y)\,dx\,dy=\int_{y=c}^{d}\int_{x=x_1(y)}^{x_2(y)} f(x,y)\,dx\,dy \longrightarrow (2)$$

Step2:- The evaluation of (2) will be the value of (1) on changing the order of integration.

This can be ~~to~~ vice versa also.

Evaluation of double integral by changing into polar coordinates

Step1:- Given a double integral with limits, we use the well known polar form of substitution $x=r\cos\theta$, $y=r\sin\theta$. This will give us $x^2+y^2=r^2$, $\frac{y}{x}=\tan\theta$ and it should be noted that

~~$\frac{dx}{d}$~~ $dx\,dy = J\,dr\,d\theta$

where J is the Jacobian of the transformation given by.

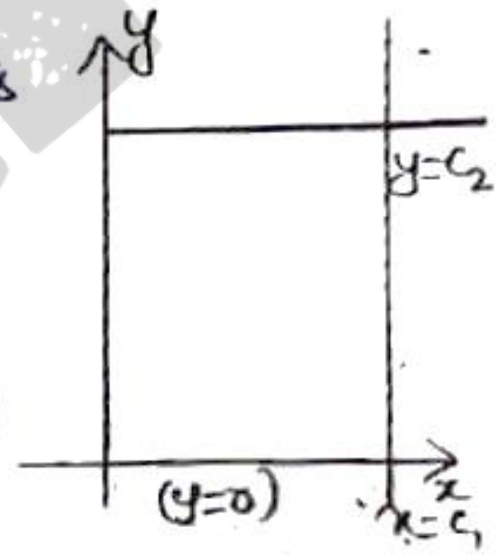
$$J=\frac{\partial(x,y)}{\partial(r,\theta)}=\begin{vmatrix}\frac{\partial x}{\partial r} & \frac{\partial x}{\partial\theta}\\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial\theta}\end{vmatrix}=\begin{vmatrix}\cos\theta & -r\sin\theta\\ \sin\theta & r\cos\theta\end{vmatrix}=r$$

Step 2:- Hence, $dx\,dy = r\,dr\,d\theta$ and we need to change the limits of integration to r, θ suitably for the purpose of evaluation.

Remark:- The method might be advantageous if the terms of the form x^2+y^2 are involved in $f(x,y)$ and terms like $\sqrt{a^2-y^2}$, $\sqrt{a^2-x^2}$ etc. are involved in limits.

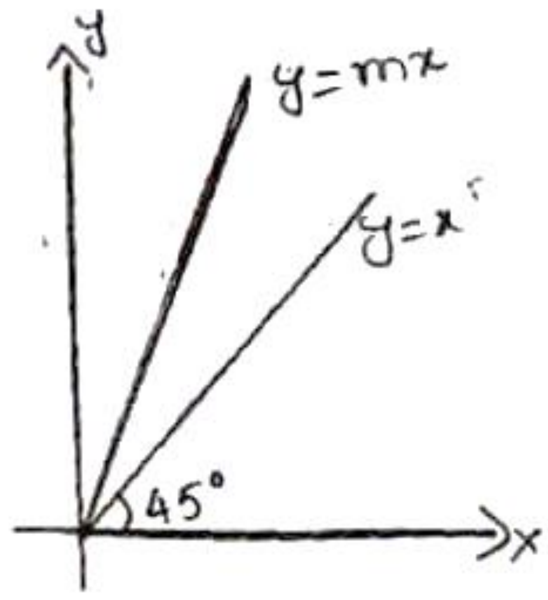
1. Straight lines:-

(i) $x=0$ and $y=0$ are respectively the equations of y and x-axis.

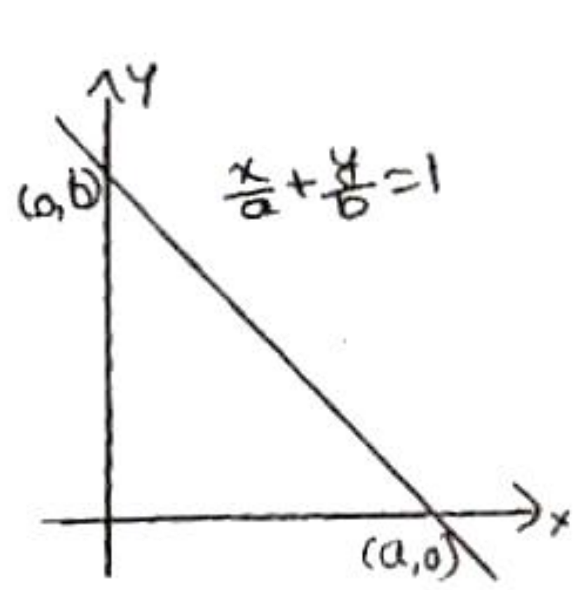


(ii) $x=c_1$ and $y=c_2$ are respectively the equations of a line parallel to y-axis and a line parallel to x-axis.

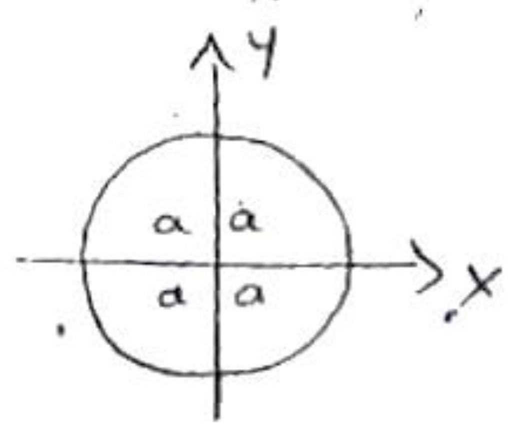
(iii) $y=mx$ is a straight line passing through the origin and in particular $y=x$ is a straight line passing through the origin subtending an angle 45° with the x-axis.



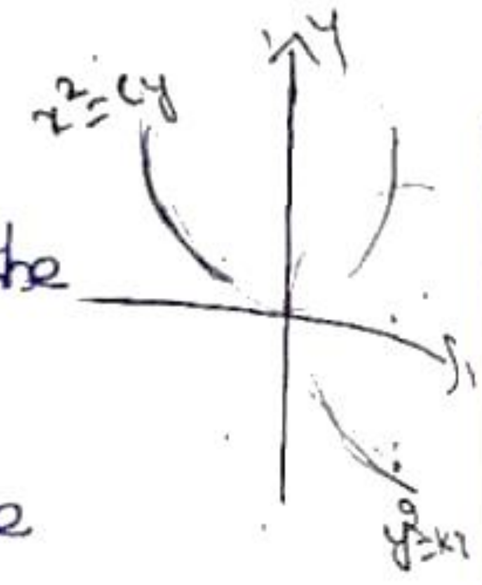
(iv) $\frac{x}{a}+\frac{y}{b}=1$ is a straight line having x intercept a and y intercept b, being a straight line passing through $(a,0)$ and $(0,b)$



2. Circle: $x^2+y^2=a^2$ is a circle with centre origin and radius a.



3. Parabola :-



$y^2 = kx$ is a parabola symmetrical about the x-axis.

$x^2 = cy$ is a parabola symmetrical about the y-axis.

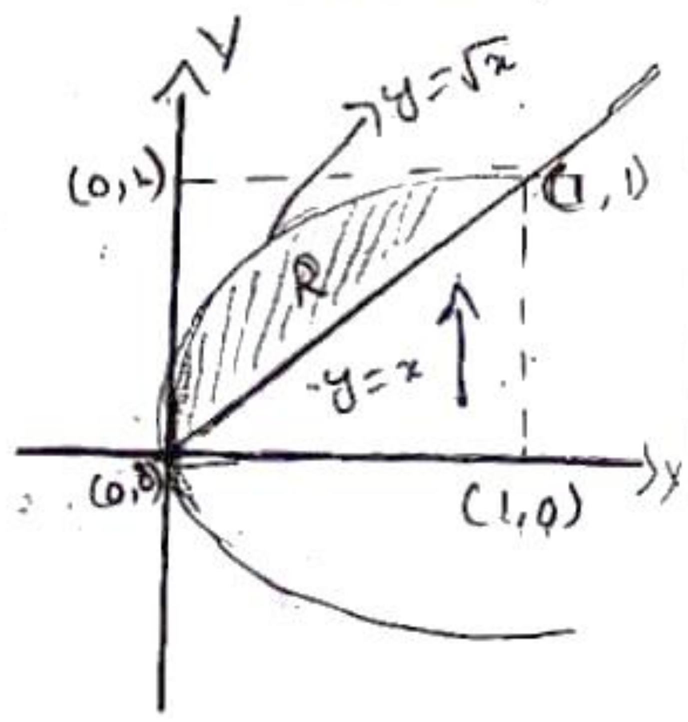
Evaluation of double integral by changing the order of integration.

1. Evaluate $\int_0^1 \int_x^{\sqrt{x}} xy \, dy \, dx$ by changing the order of integration ** June'16 Dec 18

→ Given $I = \int_{x=0}^{1} \int_{y=x}^{\sqrt{x}} xy \, dy \, dx$

$y = x$ is a straight line passing through the origin making an angle 45° with x-axis

$y = \sqrt{x}$ $[y^2 = x]$. This is a parabola symmetric about x-axis (c-shape)



Given $y = x$ and $y = \sqrt{x}$. So, $x = \sqrt{x}$

Square on B.S

$\Rightarrow x^2 = x \Rightarrow x^2 - x = 0$

$x(x-1) = 0$

$\Rightarrow x = 0, \; x - 1 = 0$

$x = 0, \; x = 1$

$x = 0, \; x = 1$
$y = 0, \; y = 1$ (0,0) and (1,1) are the points of intersection

Change of order

$y \to 0$ to 1

$x \to y^2$ to y

$I = \int_{y=0}^{1} \int_{x=y^2}^{y} xy \, dx \, dy$

$I = \int_{y=0}^{1} y \frac{x^2}{2} \Big|_{y^2}^{y} dy$

$I = \frac{1}{2} \int_0^1 y (y^2 - y^4) \, dy$

$$I = \frac{1}{2}\int_0^1 (y^3 - y^5)\,dy$$

$$I = \frac{1}{2}\left[\frac{y^4}{4} - \frac{y^6}{6}\right]_0^1$$

$$I = \frac{1}{2}\left[\frac{1}{4} - \frac{1}{6}\right]$$

$$I = \frac{1}{2}\left[\frac{6-4}{24}\right] = \frac{1}{\cancel{2}}\left[\frac{\cancel{2}}{24}\right]$$

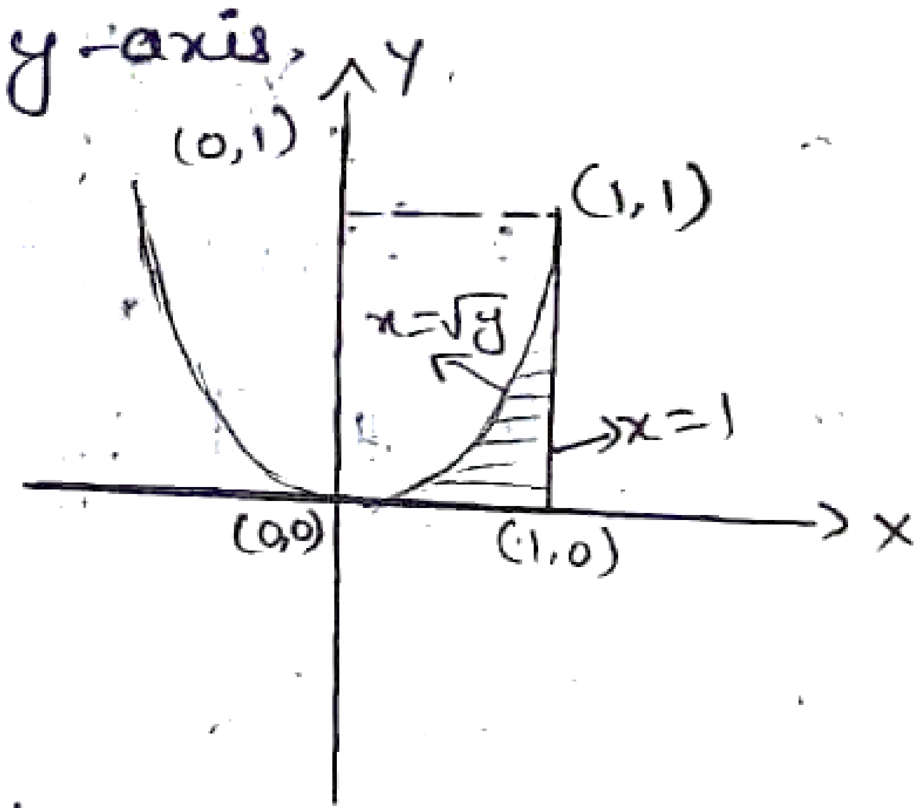
$$\boxed{I = \frac{1}{24}}$$

2. change the order of integration and hence Evaluate $\int_0^1 \int_{\sqrt{y}}^1 dx\,dy$

→ Given $I = \int_{y=0}^{1} \int_{x=\sqrt{y}}^{1} 1\,dx\,dy$

$x = \sqrt{y}$, $[x^2 = y]$ This is a parabola symmetric about y-axis

$x = 1$, It is a line parallel to y-axis.



$$I = \int_{x=0}^{1}\int_{y=0}^{x^2} 1\,dy\,dx$$

$$I = \int_0^1 y\Big|_0^{x^2} dx$$

$$I = \int_0^1 x^2\,dx$$

$$I = \frac{x^3}{3}\Big|_0^1$$

$$\boxed{I = \frac{1}{3}}$$

change of order

$y \to 0$ to 1

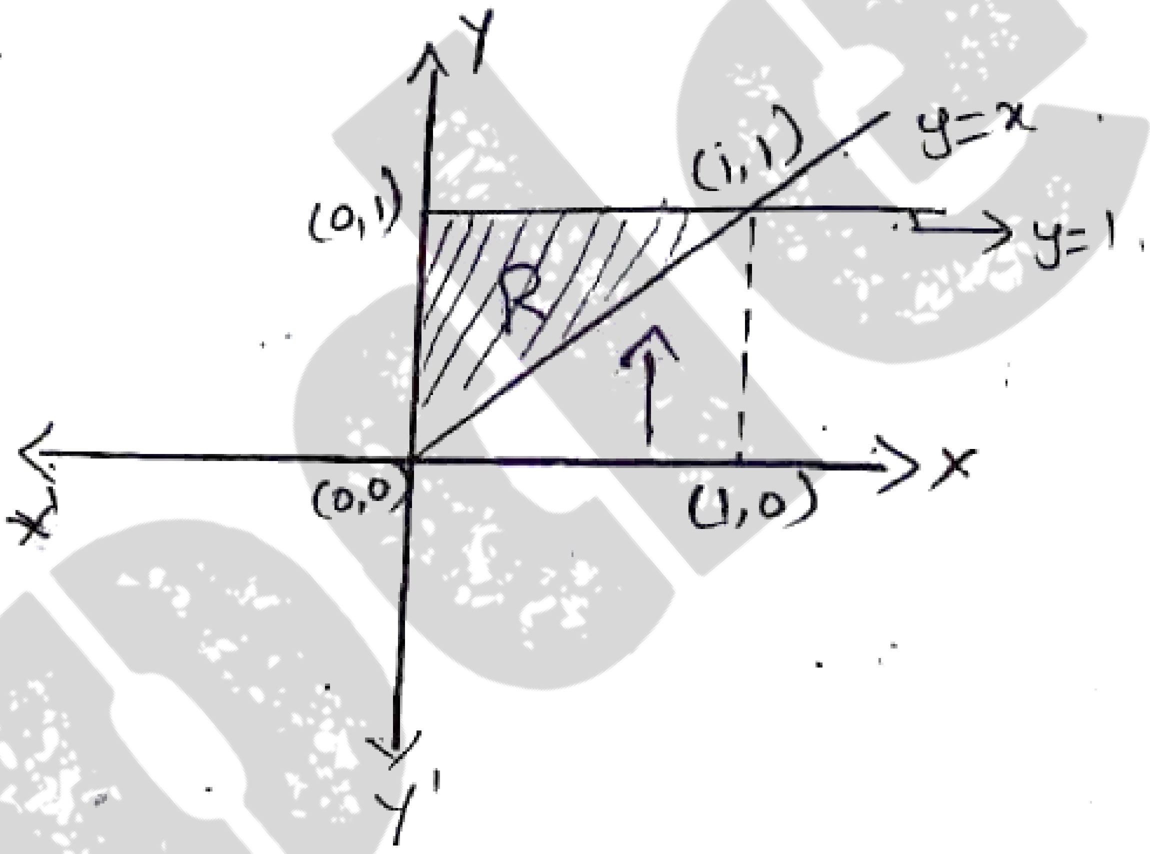
$x \to 0$ to x^2

3. Evaluate by changing ** the order of integration for (Dec16, Sep 20)

$$\int_0^1\int_x^1 \frac{x}{\sqrt{x^2+y^2}}\,dx\,dy$$

→ Given $$I = \int_{x=0}^{1}\int_{y=x}^{1} \frac{x}{\sqrt{x^2+y^2}}\,dy\,dx$$

$y = x$ is a straight line passing through origin making an angle $45°$ with x-axis



change of order

$y \to 0$ to 1

$x \to 0$ to y

$$I = \int_{y=0}^{1}\int_{x=0}^{y} \frac{x}{\sqrt{x^2+y^2}}\,dx\,dy$$

put $x^2+y^2 = t$

$2x\,dx = dt$

$dx = \frac{dt}{2x}$

$$I = \int_{y=0}^{1}\int_{x=0}^{y} \frac{x}{\sqrt{t}}\,\frac{dt}{2x}\,dy$$

$$I = \frac{1}{2}\int_0^1 \left[\frac{t^{-1/2+1}}{-1/2+1}\right]_0^y dy$$

$$I = \frac{1}{2}\int_0^1 \left[\frac{\sqrt{t}}{1/2}\right] dy$$

$$I = \int_0^1 \left[\sqrt{x^2+y^2}\right]_0^y dy$$

$$I = \int_0^1 \left(\sqrt{2y^2} - \sqrt{y^2}\right) dy$$

$$I = \int_0^1 \sqrt{2}\,y - y\,dy$$

$$I = \int_0^1 y(\sqrt{2}-1)\,dy$$

$$I = (\sqrt{2}-1)\left.\frac{y^2}{2}\right|_0^1$$

$$I = \frac{\sqrt{2}-1}{2}(1-0)$$

$$\boxed{I = \frac{\sqrt{2}-1}{2}}$$

Dec16, Sep20 ** *

4. change the order of integration and evaluate
$$\int_0^{4a}\int_{x^2/4a}^{2\sqrt{ax}} xy\,dy\,dx$$

$\rightarrow$ $$I=\int_{x=0}^{4a}\int_{y=\frac{x^2}{4a}}^{2\sqrt{ax}} xy\,dy\,dx$$

$y=\frac{x^2}{4a} \Rightarrow x^2=4ay \rightarrow$ It is a parabola symmetric about y-axis.

$y=2\sqrt{ax} \Rightarrow y^2=4ax \rightarrow$ It is a parabola symmetric about x-axis.

Given: $y=\frac{x^2}{4a}$ and $y=2\sqrt{ax}$

$2\sqrt{ax}=\frac{x^2}{4a}$ Square on B.S

$4ax=\frac{x^4}{16a^2} \Rightarrow 64a^3x=x^4$

$\Rightarrow x^4-64a^3x=0$

$\Rightarrow x(x^3-64a^3)=0$

$\Rightarrow \boxed{x=0}$, $x^3-64a^3=0$

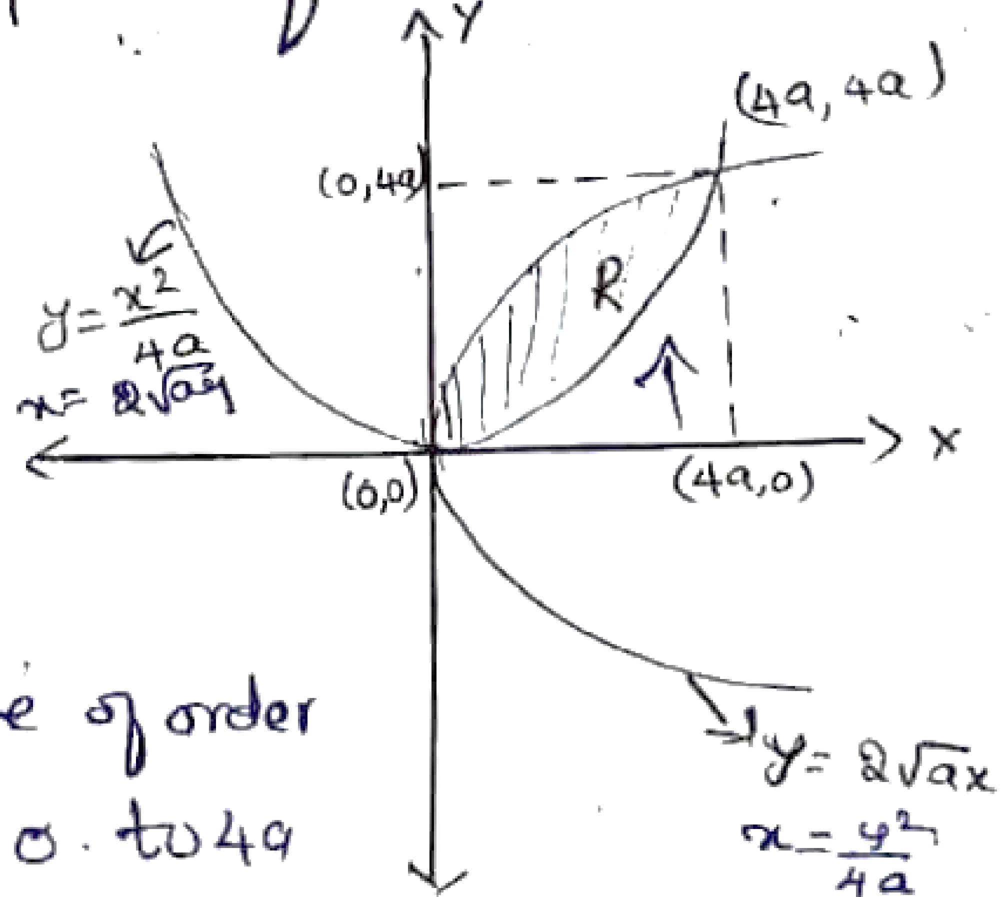
$x^3=64a^3$

$\boxed{x=4a}$

$x=0$, $y=0$

$x=4a$, $y=4a$

$(0,0)$ and $(4a,4a)$ are the point of intersection



$$I=\int_{y=0}^{4a}\int_{x=\frac{y^2}{4a}}^{2\sqrt{ay}} xy\,dx\,dy$$

$$I=\int_0^{4a} y\left.\frac{x^2}{2}\right|_{y^2/4a}^{2\sqrt{ay}} dy$$

$$I=\frac{1}{2}\int_0^{4a} y\left[4ay-\frac{y^4}{16a^2}\right]dy$$

change of order

$y\rightarrow 0$ to $4a$

$x\rightarrow \frac{y^2}{4a}$ to $2\sqrt{ay}$

$$I = \frac{1}{2}\int_0^{4a}\left(4ay^2 - \frac{y^5}{16a^2}\right)dy$$

$$I = \frac{1}{2}\left[4a\frac{y^3}{3} - \frac{1}{16a^2}\frac{y^6}{6}\right]_0^{4a}$$

$$I = \frac{1}{2}\left[\frac{4a(4a)^3}{3} - \frac{1}{96a^2}(4a)^6\right]$$

$$I = \frac{1}{2}\left[\frac{4a(64a^3)}{3} - \frac{1}{96\cancel{a^2}}(4096\,a^{\cancel{6}})\right]$$

$$I = \frac{1}{2}\left[\frac{256a^4}{3} - \frac{\overset{128}{\cancel{4096}}\,a^4}{\underset{3}{\cancel{96}}}\right]$$

$$I = \frac{1}{2}\left[\frac{256a^4 - 128a^4}{3}\right]$$

$$I = \frac{1}{\cancel{2}}\left[\frac{\overset{64}{\cancel{128}}\,a^4}{3}\right] = \frac{64a^4}{3}$$

$$\boxed{I = \frac{64a^4}{3}}$$

5. Change the order of integration and hence evaluate

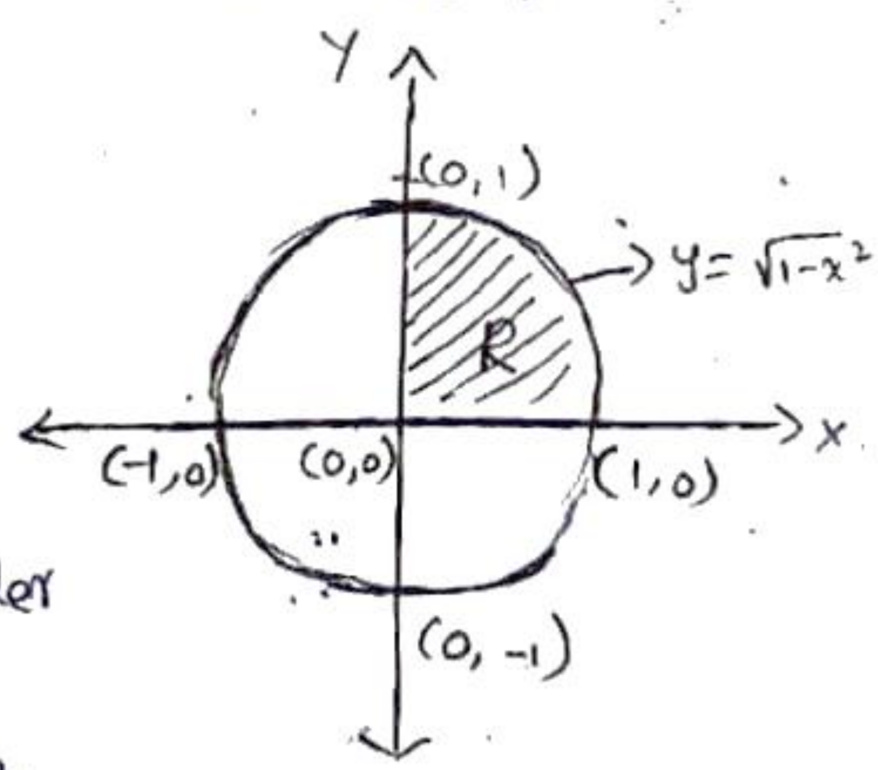
$$\int_0^1\int_0^{\sqrt{1-x^2}} y^2\,dx\,dy$$

$\rightarrow$ $$I = \int_{x=0}^{1}\int_{y=0}^{\sqrt{1-x^2}} y^2\,dy\,dx$$

$y = 0 \Rightarrow x$-axis

$y = \sqrt{1-x^2} \Rightarrow y^2 = 1 - x^2 \Rightarrow x^2 + y^2 = 1^2$

It is a circle with centre $(0,0)$ and $r = 1$



$$I = \int_{y=0}^{1}\int_{x=0}^{\sqrt{1-y^2}} y^2\,dx\,dy$$

$$I = \int_{y=0}^{1}\left[x\,y^2\right]_0^{\sqrt{1-y^2}} dy$$

change of order
$y \rightarrow 0$ to 1
$x \rightarrow 0$ to $\sqrt{1-y^2}$

$$I = \int_0^1 y^2\left(\sqrt{1-y^2}\right)dy$$

put $y = \sin\theta$

$dy = \cos\theta \, d\theta$

$y = 0, \quad \theta = 0$

$y = 1, \quad \theta = \pi/2$

$$I = \int_{\theta=0}^{\pi/2} \sin^2\theta \left(\sqrt{1-\sin^2\theta}\right) \cos\theta \, d\theta \qquad (\because 1-\sin^2\theta = \cos^2\theta)$$

$$I = \int_{\theta=0}^{\pi/2} \sin^2\theta \, \cos^2\theta \, d\theta$$

$$\left[\int_0^{\pi/2} \sin^m\theta \, \cos^n\theta \, d\theta = \left(\frac{(m-1)(m-3)\ldots \quad (n-1)(n-3)\ldots}{(m+n)\,(m+n-2)\,\,(m+n-4)\ldots}\right) K\right]$$

$\Rightarrow$ m n

odd odd
odd even
even odd
$\Bigg\}$ $K = 1$

$\Rightarrow$ Both m and n are even. hence $K = \frac{\pi}{2}$

even, even $\rightarrow$ $K = \frac{\pi}{2}$

$$= \frac{(2-1)(2-1)}{(2+2)(2+2-2)} \times \frac{\pi}{2}$$

$$\Rightarrow \frac{(1)(1)}{4(2)} \times \frac{\pi}{2}$$

$$= \frac{\pi}{16}$$

* * 2018 June, 2021 sep

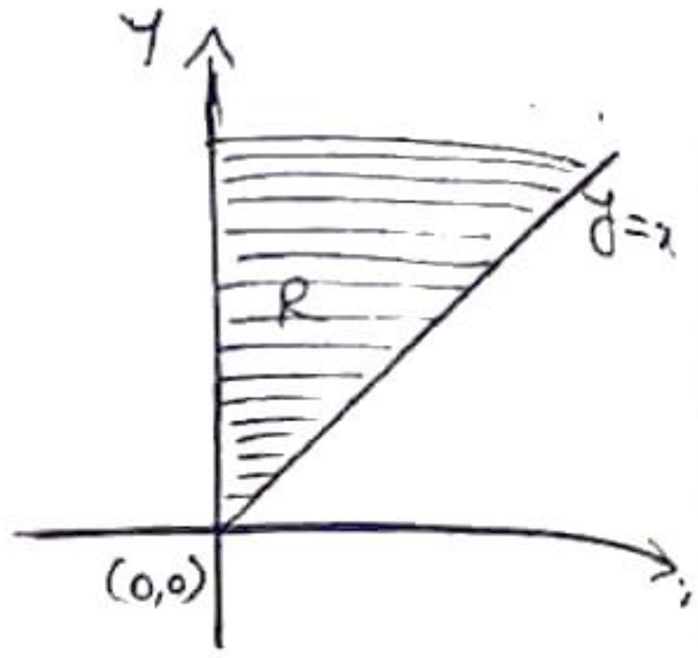
6. Change the order of integration and hence evaluate

$$\int_0^\infty \int_x^\infty \frac{e^{-y}}{y} \, dy \, dx$$

$\rightarrow$ $$I = \int_{x=0}^{\infty} \int_{y=x}^{\infty} \frac{e^{-y}}{y} \, dy \, dx$$

$y = x \Rightarrow$ straight line passing through origin making an angle $45°$ with x-axis.

$$I = \int_{y=0}^{\infty} \int_{x=0}^{y} \frac{e^{-y}}{y}\, dx\, dy$$



$$I = \int_{0}^{\infty} \frac{e^{-y}}{y}\, x \Big]_{0}^{y} dy$$

$$I = \int_{0}^{\infty} \frac{e^{-y}}{\cancel{y}} (\cancel{y})\, dy$$

Change of order
$y \to 0$ to ∞
$x \to 0$ to y

$$I = \int_{0}^{\infty} e^{-y}\, dy$$

$$I = \frac{e^{-y}}{-1}\Big]_{0}^{\infty}$$

$$I = -(e^{-\infty} - e^{-0}) = -(0-1)$$

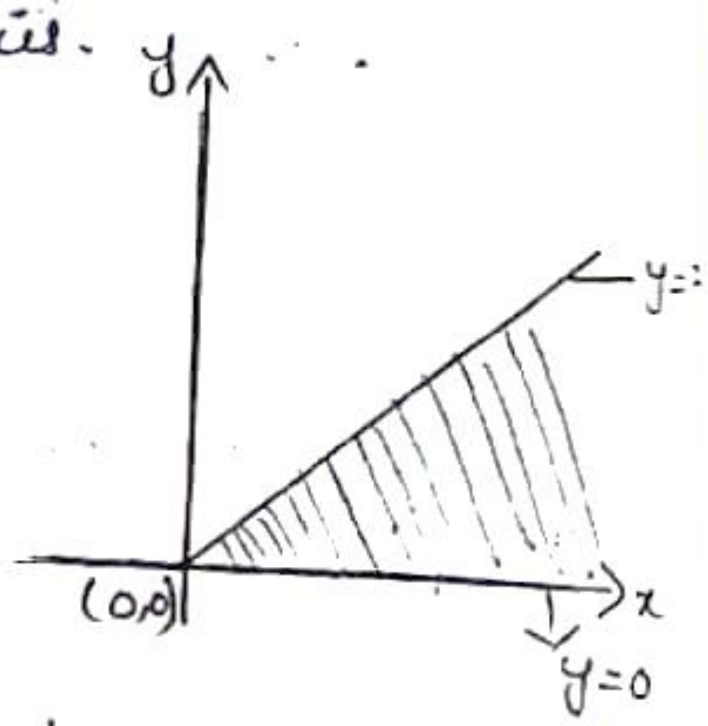
$$\boxed{I = 1}$$

7. H.W Evaluate $\int_{0}^{\infty}\int_{0}^{x} x e^{-x^2/y}\, dy\, dx$ by changing the order of integration.

→ $I = \int_{x=0}^{\infty}\int_{y=0}^{x} x e^{-x^2/y}\, dy\, dx$

$y=0 \Rightarrow x$-axis

$y=x$ is a straight line passing through origin making an angle 45° with x-axis.



$$I = \int_{y=0}^{\infty}\int_{x=y}^{\infty} x e^{-x^2/y}\, dx\, dy$$

Put $\frac{x^2}{y} = t$

$\frac{2x}{y} dx = dt$

$x\, dx = \frac{y}{2} dt$

change of order
$y \to 0$ to ∞
$x \to y$ to ∞

$x=y \quad t=y$

$x=\infty \quad t=\infty$

$$= \int_{y=0}^{\infty}\int_{t=y}^{\infty} \frac{y}{2}\, dt\, e^{-t}\, dy$$

$$= \frac{1}{2}\int_{y=0}^{\infty} y\, \frac{e^{-t}}{-1}\Big]_{y}^{\infty} dy$$

$$= \frac{1}{2}\int_{y=0}^{\infty} \underset{u}{y}\,\underset{v}{e^{-y}}\,dy$$

$$I = \frac{1}{2}\left[y\,\frac{e^{-y}}{-1}\Big]_0^{\infty} - (1)\frac{e^{-y}}{-1}\right]_0^{\infty}$$

$$I = \frac{1}{2}\left[-(\cancel{\infty e^{-\infty}} - 0) - (e^{-\infty} - e^{-0})\right]$$

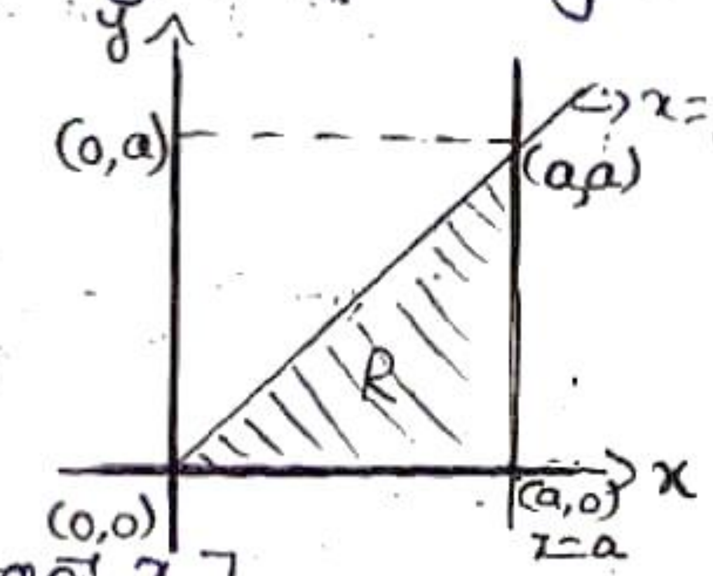
$$I = \frac{1}{2}(1)$$

$$\boxed{I = \frac{1}{2}}$$

8. Evaluate $\int_0^a \int_y^a \frac{x}{x^2+y^2}\,dx\,dy$ by changing the order of integration

$\rightarrow$ $$I = \int_{y=0}^{a}\int_{x=y}^{a} \frac{x}{x^2+y^2}\,dx\,dy$$

$x=y \Rightarrow$ Straight line passing through orgin making an angle 45° with x-axis.



$x=a \Rightarrow$ line parallel to y-axis

$$I = \int_{x=0}^{a}\int_{y=0}^{x} \frac{x}{x^2+y^2}\,dy\,dx \qquad \left[\because \int \frac{1}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}\right]$$

Change of order

$y \rightarrow 0$ to a

$x \rightarrow 0$ to x

$$I = \int_{x=0}^{a} \left[\cancel{x}\frac{1}{\cancel{x}}\tan^{-1}\frac{y}{x}\right]_0^{x} dx$$

$$I = \int_0^a \left[\tan^{-1}(1) - \tan^{-1}(0)\right] dx$$

$$I = \int_0^a \frac{\pi}{4}\,dx$$

$$= \frac{\pi}{4}\,x\Big]_0^a$$

$$\boxed{I = \frac{\pi a}{4}}$$

9. →2019 Dec Evaluate $\int_{-2}^{2}\int_{0}^{\sqrt{4-x^2}} (2-x)\,dx\,dy$ by changing the order.

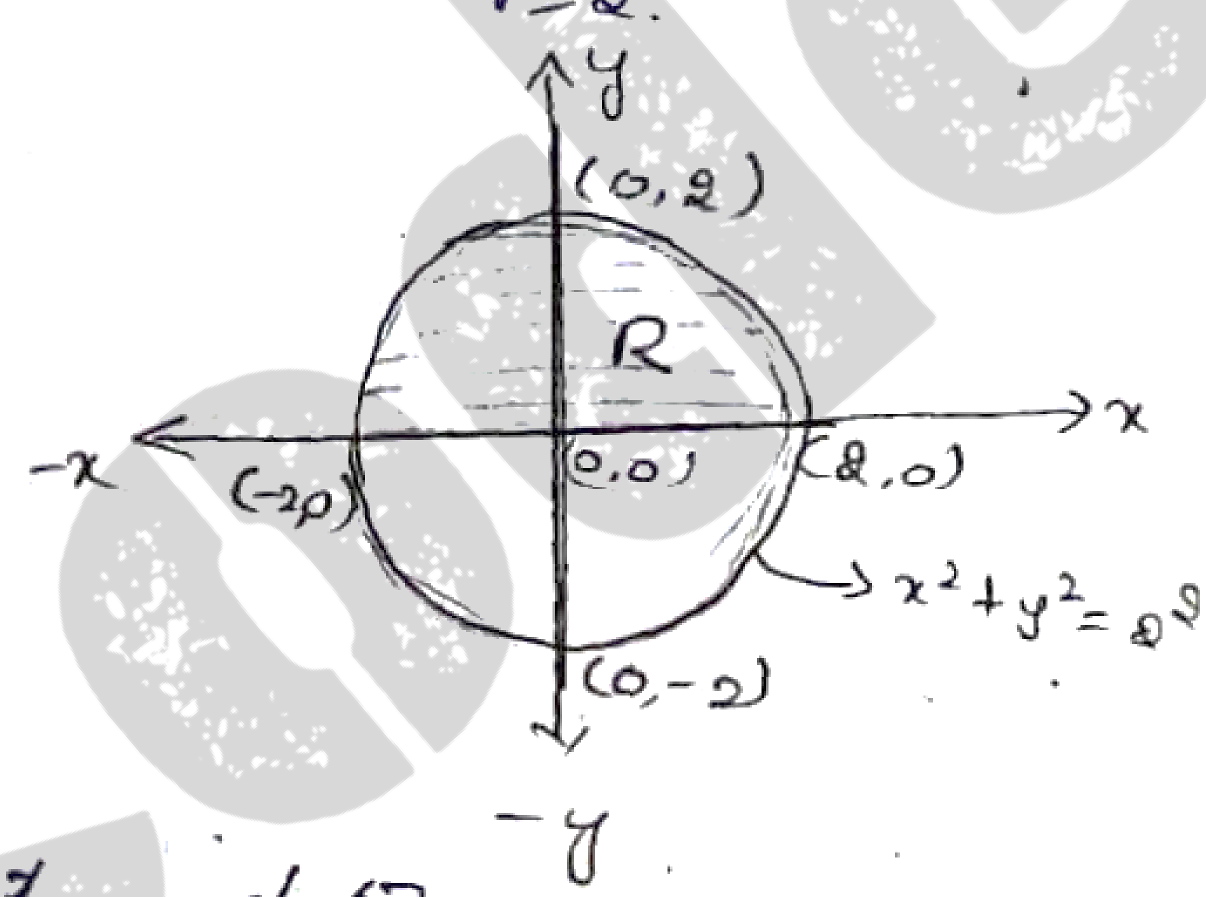
→ $I = \int_{x=-2}^{2}\int_{y=0}^{\sqrt{4-x^2}} (2-x)\,dy\,dx$

$y=0 \Rightarrow x-\text{axis}$

$y=\sqrt{4-x^2} \Rightarrow y^2 = 4-x^2$

$y^2 = 2^2 - x^2$ It is circle with centre $(0,0)$ and $r=2$.

$x^2+y^2 = 2^2$



$I = \int_{y=0}^{2}\int_{x=-\sqrt{4-y^2}}^{\sqrt{4-y^2}} (2-x)\,dx\,dy$

$I = \int_{y=0}^{2} \left[2x - \frac{x^2}{2}\right]_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} dy$

$I = \int_{y=0}^{2} 2\left[\sqrt{4-y^2}+\sqrt{4-y^2}\right] - \frac{1}{2}\left[\cancel{(\sqrt{4-y^2})^2} - \cancel{(\sqrt{4-y^2})^2}\right] dy$

$I = 2\int_{0}^{2} 2\left[\sqrt{4-y^2}\right] \cancel{-\frac{4}{2}}\, dy$

$I = 4\int_{0}^{2} \sqrt{2^2-y^2}\,dy$ $\left[\int\sqrt{a^2-x^2} = \frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right]$

$I = 4\left[\cancel{\frac{y\sqrt{4-y^2}}{2}}^{\,0} + \frac{2^2}{2}\sin^{-1}\left(\frac{y}{2}\right)\right]_0^2$

$I = 4\left[\frac{4}{2}\left(\sin^{-1}(1) - \sin^{-1}(0)\right)\right]$

$I = \frac{\cancel{16}^{\,\cancel{8}\,4}}{\cancel{2}}\left[\frac{\pi}{\cancel{2}} - 0\right]$

$\boxed{I = 4\pi}$

10. Evaluate $\int_{-2}^{2}$ S.T $\int_0^{2a}\int_0^{\sqrt{2ax-x^2}} f(x,y)\,dy\,dx = \int_0^{a}\int_{a-\sqrt{a^2-y^2}}^{a+\sqrt{a^2-y^2}} f(x,y)\,dx\,dy$

$\rightarrow$ $I = \int_{x=0}^{2a}\int_{y=0}^{\sqrt{2ax-x^2}} f(x,y)\,dy\,dx$

$y=0 \Rightarrow x-\text{axis}$

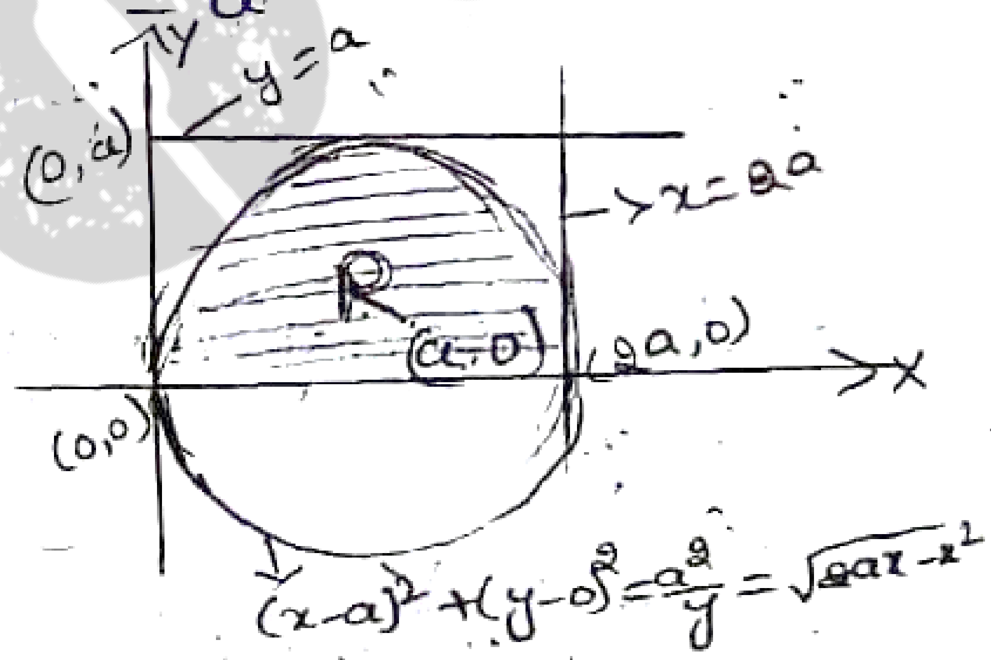
$y=\sqrt{2ax-x^2}$

$y^2 = 2ax - x^2$

$x^2+y^2 = 2ax = (x-a)^2+(y-0)^2 = a^2$

It is a circle with $C(a,0)$ and $r=a$. It is evident that the circle passes through origin having the centre on x-axis, and radius $=a$

$$\boxed{I = \int_{y=0}^{a}\int_{x=a-\sqrt{a^2-y^2}}^{a+\sqrt{a^2-y^2}} f(x,y)\,dx\,dy = RHS}$$



$(x-a)^2+(y^2-0)=a^2$

$(x-a)^2 = a^2-y^2$

$x-a = \pm\sqrt{a^2-y^2}$

$x = a\pm\sqrt{a^2-y^2}$

* June 2018

11. Evaluate $\int_0^1\int_{x^2}^{2-x} xy\,dy\,dx$ by changing the order of integration

$\rightarrow$ $I = \int_{x=0}^{1}\int_{y=x^2}^{2-x} xy\,dy\,dx$

$y=x^2 \Rightarrow$ It is a parabola symmetric about y-axis

$y=2-x \Rightarrow x+y=2 \Rightarrow \frac{x}{2}+\frac{y}{2}=1$

$\Rightarrow$ It is straight line

Passing through points $(2,0)$ and $(0,2)$

$y = x^2$ and $y = 2-x$

$2 - x = x^2$

$x^2 + x - 2 = 0$

$x^2 + 2x - x - 2 = 0$

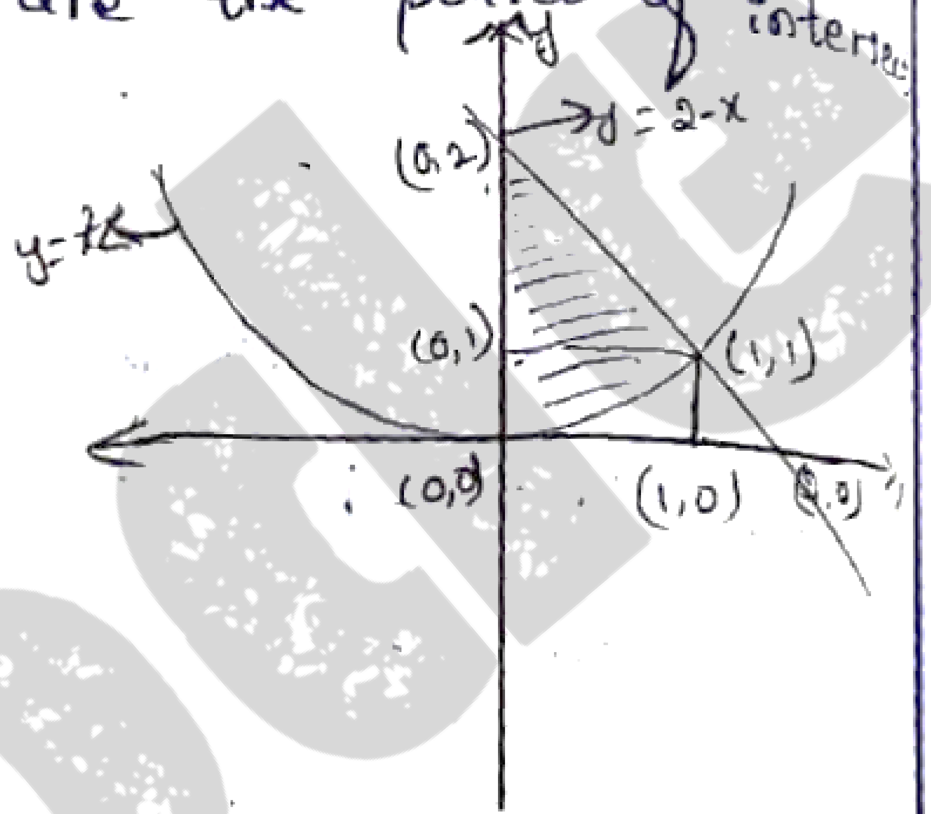
$x(x+2) - 1(x+2) = 0$

$\boxed{x=1}$ or $\boxed{x=-2}$

$x = 1 \quad y = 1$

$x = -2 \quad y = 4$

$(1,1)$ and $(-2,4)$ are the point of intersection



$$I = \int_{y=0}^{1}\int_{x=0}^{\sqrt{y}} xy \, dx\,dy + \int_{y=1}^{2}\int_{x=0}^{2-y} xy \, dx\,dy$$

$$I = \int_{y=0}^{1} \frac{x^2}{2} y \Big]_0^{\sqrt{y}} dy + \int_{y=1}^{2} \left[\frac{x^2}{2} y\right]_0^{2-y} dy$$

$$= \frac{1}{2}\int_{y=0}^{1} (\sqrt{y})^2 y \, dy + \frac{1}{2}\int_{y=1}^{2} (2-y)^2 y \, dy$$

$(2-y)^2 = 4 + y^2 - 4y$
$= 4y + y^3 - 4y^2$

$$= \frac{1}{2}\int_{y=0}^{1} y^2 \, dy + \frac{1}{2}\int_{y=1}^{2} 4y + y^3 - 4y^2 \, dy$$

$$= \frac{1}{2} \frac{y^3}{3}\Big]_0^1 + \frac{1}{2}\left[\frac{\cancel{4}^{2} y^2}{\cancel{2}} + \frac{y^4}{4} - \frac{4y^3}{3}\right]_1^2$$

$$= \frac{1}{2}\left(\frac{1}{3}\right) + \frac{1}{2}\left[\left(2(2)^2 + \frac{1}{4}(2)^4 - \frac{4}{3}(2^3)\right) - \left(2 + \frac{1}{4} - \frac{4}{3}\right)\right]$$

$$[2(4-1) + \frac{1}{4}(16-1) - \frac{4}{3}(8-1)]$$

$$= \frac{1}{6} + \frac{1}{2}\left[\cancel{8 + \frac{16}{4} - \frac{32}{3} - 2 - \frac{1}{4} - \frac{4}{3}}\right]$$

$10 - \frac{32}{3} - \frac{1}{4} - \frac{4}{3}$

$10 - \frac{128 - 3 - 16}{12}$

$$= \frac{1}{6} + \frac{1}{2}\left[\frac{72 + 45 - 112}{12}\right] =$$

$\frac{5}{24}$

$$= \frac{1}{6} + \frac{5}{24} = \frac{4+5}{24} = \frac{\cancel{9}}{\cancel{24}} = \frac{3}{8}$$

$\boxed{I = 3/8}$

12. change the order of integration, Evaluate $\int_0^1 \int_{\sqrt{y}}^{2-y} xy \, dx \, dy$

$\rightarrow I = \int_{y=0}^{1} \int_{x=\sqrt{y}}^{2-y} xy \; dx \, dy$

$x=\sqrt{y} \Rightarrow x^2 = y \Rightarrow$ parabola symmetric about y-axis

$x = 2-y \Rightarrow x+y=2 \Rightarrow \frac{x}{2}+\frac{y}{2}=1 \Rightarrow$ straight line passing through $(2,0)$ and $(0,2)$

$x=\sqrt{y}$ and $x = 2-y \Rightarrow \sqrt{y} = 2-y$

$$(2-y)^2 = y$$

$$4+y^2-4y = y$$

$$y^2-4y-y+4=0 \Rightarrow y(y-4)-1(y-4)=0$$

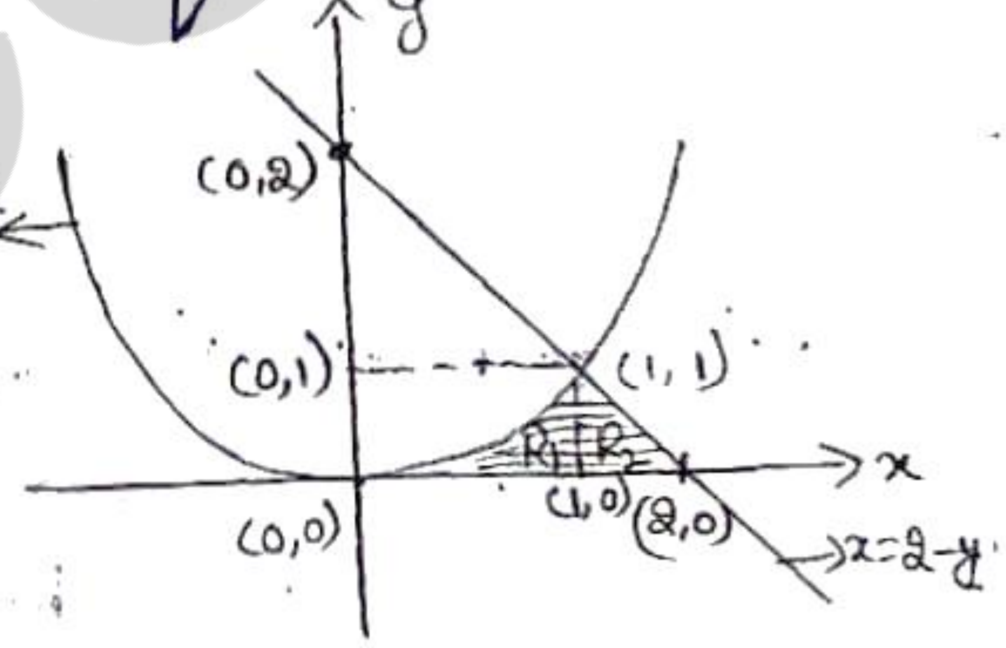
$$\cancel{y^2-5y+4=0} \qquad (y-4)(y-1)=0$$

$$\boxed{y=4} \quad \boxed{y=1}$$

$$\boxed{x=\pm 2} \quad \boxed{x=\pm 1}$$

$(1,1)\,(-1,1)\,(2,4),\,(-2,4)$ are the points of intersection.

$$I = \int_{x=0}^{1}\int_{y=0}^{x^2} xy \, dy\,dx + \int_{x=1}^{2}\int_{y=0}^{2-x} xy \; dy \; dx$$



$$I = \int_0^1 x\frac{y^2}{2}\Big|_0^{x^2} dx + \int_{x=1}^{2} x\frac{y^2}{2}\Big|_0^{2-x} dx$$

$$I = \frac{1}{2}\int_0^1 [x^5]\,dx + \frac{1}{2}\int_1^2 [x(2-x)^2]\,dx$$

$$(4+x^2-4x)$$

on changing the order

I $x \rightarrow 0$ to 1

$y \rightarrow 0$ to x^2

II $x \rightarrow 1$ to 2

$y \rightarrow 0$ to $2-x$

$$I = \frac{1}{2}\left[\frac{x^6}{6}\right]_0^1 + \frac{1}{2}\left[4x+x^3-4x^2\right]_1^2$$

$$I = \frac{1}{12} + \frac{1}{2}\left[\frac{\cancel{4}^{2}x^2}{\cancel{2}} + \frac{x^4}{4} - 4\frac{x^3}{3}\right]_1^2$$

$$I = \frac{1}{12} + \frac{1}{2}\left[2(4-1) + \frac{1}{4}(16-1) - \frac{4}{3}(8-1)\right]$$

$$I = \frac{1}{12} + \frac{1}{2}\left[6 + \frac{15}{4} - \frac{28}{3}\right]$$

$$I = \frac{1}{12} + \frac{5}{24}$$

$$\boxed{I = \frac{7}{24}}$$

13. Change the order of integration $\int_0^1 \int_x^{\sqrt{2-x^2}} \frac{x}{\sqrt{x^2+y^2}}\, dx\, dy$

$\rightarrow$ $I = \int_{x=0}^{1} \int_{y=x}^{\sqrt{2-x^2}} \frac{x}{\sqrt{x^2+y^2}}\, dy\, dx$

$y = x \Rightarrow$ Straight line passing through origin making angle 45° about x-axis.

$y = \sqrt{2-x^2} \Rightarrow y^2 = 2-x^2 \Rightarrow$

$x^2+y^2 = (\sqrt{2})^2$

$\Rightarrow$ Circle with centre $(0,0)$ and $r = \sqrt{2}$

$y = x$ and $y = \sqrt{2-x^2} \Rightarrow \sqrt{2-x^2} = x$

$\Rightarrow 2 - x^2 = x^2$

$\Rightarrow \cancel{2}x^2 = \cancel{2}$

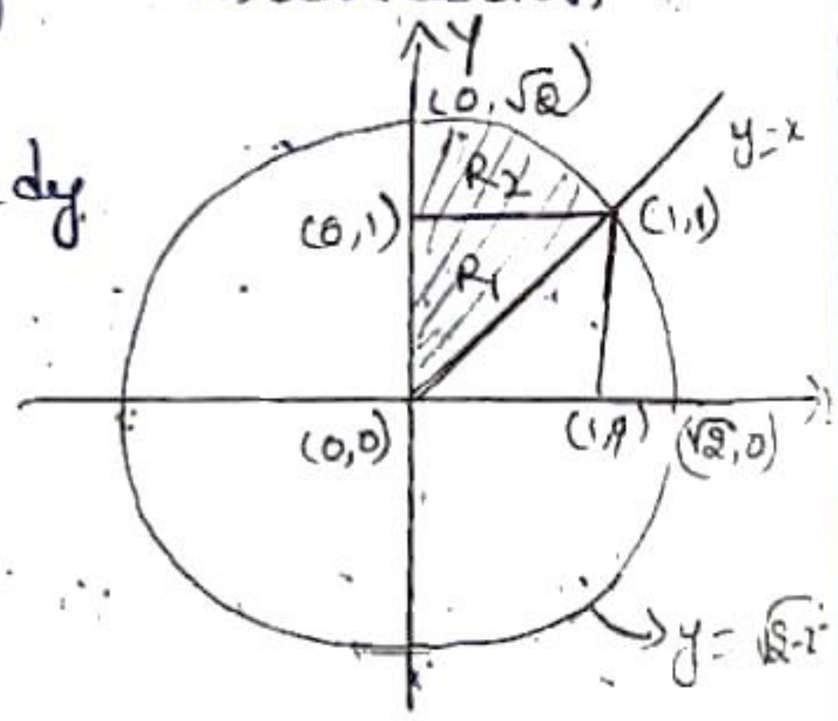
$\Rightarrow x^2 = 1$

$\Rightarrow x = \pm 1$

$\boxed{x=1}$ $\boxed{y=1}$

$\boxed{x=-1}$ $\boxed{y=-1}$

$(1,1)$ and $(-1,-1)$ are the point of intersection



$$I = \int_{y=0}^{1} \int_{x=0}^{x=y} \frac{x}{\sqrt{x^2+y^2}}\, dx\, dy + \int_{y=1}^{\sqrt{2}} \int_{x=0}^{\sqrt{2-y^2}} \frac{x}{\sqrt{x^2+y^2}}\, dx\, dy$$

put $x^2+y^2 = t$

$2x\, dx = dt$

$dx = \frac{dt}{2x}$

$$I = \int_{y=0}^{1} \int_{x=0}^{y} \frac{x}{\sqrt{t}} \frac{dt}{2x}\, dy + \int_{y=1}^{\sqrt{2}} \int_{x=0}^{\sqrt{2-y^2}} \frac{x}{\sqrt{t}} \frac{dx}{2\cancel{x}}\, dy$$

$$I = \cancel{\frac{1}{2}} \int_0^1 \frac{t^{-1/2+1}}{\cancel{-1/2+1}}\, dy + \cancel{\frac{1}{2}} \int_{\cancel{0}}^{\sqrt{2}} \frac{t^{-1/2+1}}{\cancel{-1/2+1}}\, dy$$

$$I = \int_0^1 \sqrt{x^2+y^2}\Big]_0^y dy + \int_1^{\sqrt{2}} \sqrt{x^2+y^2}\Big]_0^{\sqrt{2-y^2}} dy$$

$$I = \int_0^1 \left(\sqrt{2y^2} - \sqrt{y^2}\right) dy + \int_{y=1}^{\sqrt{2}} (\sqrt{2} - y)\, dy$$

$$I = \int_0^1 (\sqrt{2}\cdot y - y)\, dy + \int_1^{\sqrt{2}} (\sqrt{2} - y)\, dy$$

$$I = \int_0^1 (\sqrt{2} - 1) y\, dy + \left[\sqrt{2}\, y - \frac{y^2}{2}\right]_1^{\sqrt{2}}$$

$$I = (\sqrt{2}-1) \left.\frac{y^2}{2}\right|_0^1 + 2 - \sqrt{2} - \frac{1}{2}$$

$$I = (\sqrt{2}-1)\left(\frac{1}{2} - 0\right) + \frac{3}{2} - \sqrt{2}$$

$$I = \frac{\sqrt{2}-1}{2} + \frac{3}{2} - \frac{\sqrt{2}}{1}$$

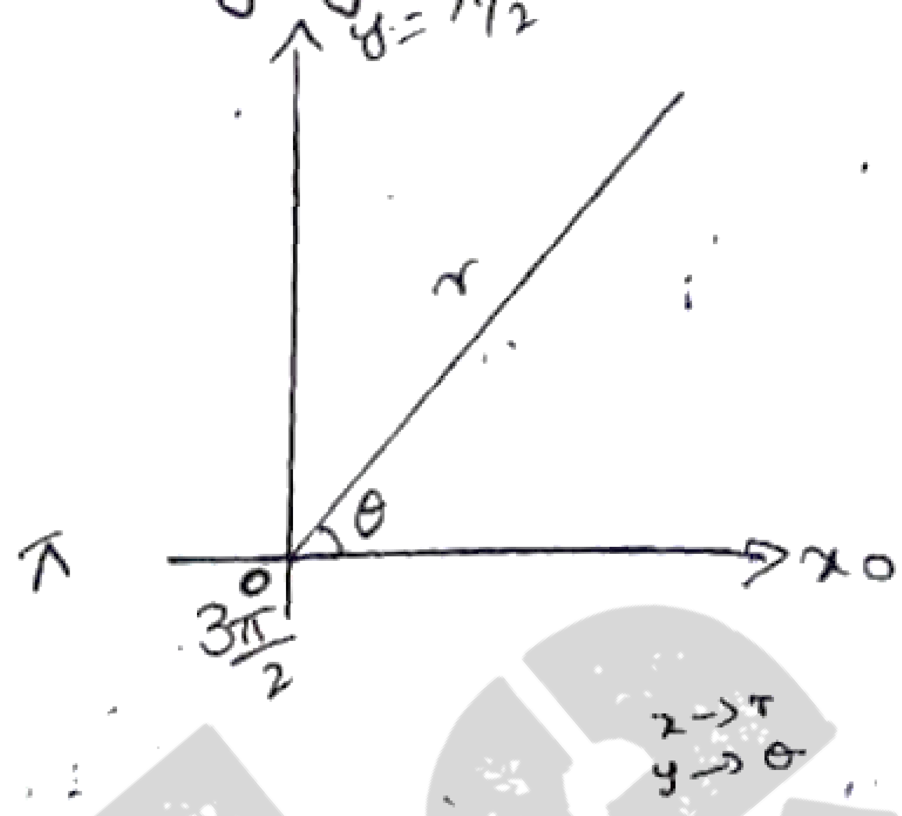
$$I = \frac{1}{\sqrt{2}} - \frac{1}{2} + \frac{3}{2} - \frac{\sqrt{2}}{1}$$

$$= 1 - \frac{1}{\sqrt{2}}$$

$$\boxed{I = \frac{\sqrt{2}-1}{\sqrt{2}}}$$

Evaluation by changing into polars :-

1. Evaluate $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\, dx\, dy$ by changing into polar co-ordinates.



$\to$ $$I = \int_{x=0}^{\infty} \int_{y=0}^{\infty} e^{-(x^2+y^2)}\, dy\, dx$$

In polar we have $x = r\cos\theta$

$y = r\sin\theta$

$$x^2+y^2 = r^2$$

$$\Rightarrow dx\, dy = r\, dr\, d\theta$$

$$\theta \to 0 \text{ to } \pi/2$$

Since, x, y varies from 0 to ∞, r also varies from 0 to ∞

$$\therefore \quad I = \int_{\theta=0}^{\pi/2} \int_{r=0}^{\infty} e^{-r^2} r\, dr\, d\theta$$

put $r^2 = t$ $\qquad r=0 \quad t=0 \qquad dr = \frac{dt}{2}$

$2r\, dr = dt \qquad r=\infty \quad t=\infty$

$$I = \int_{\theta=0}^{\pi/2} \int_{t=0}^{\infty} e^{-t} \frac{dt}{2}\, d\theta$$

$$I = \frac{1}{2} \int_0^{\pi/2} \left. \frac{e^{-t}}{-1} \right|_0^\infty d\theta$$

$$I = -\frac{1}{2} \int_0^{\pi/2} [\cancel{e^{-\infty}}^{\,0} - e^{-0}]\, d\theta$$

$$I = \frac{1}{2} \int_0^{\pi/2} d\theta$$

$$I = \frac{1}{2}\, \theta \Big|_0^{\pi/2}$$

$$I = \frac{1}{2}\left[\frac{\pi}{2} - 0\right] = \frac{\pi}{4}$$

$$\boxed{I = \frac{\pi}{4}}$$

2. change the integral $\int_{-a}^{a}\int_{0}^{\sqrt{a^2-x^2}} \sqrt{x^2+y^2}\, dy\, dx$

$\rightarrow$ $I = \int_{x=-a}^{a}\int_{y=0}^{\sqrt{a^2-x^2}} \sqrt{x^2+y^2}\, dy\, dx$

In polars, $x = r\cos\theta$, $y = r\sin\theta$., $x^2+y^2=r^2$

$dx\, dy = r\, dr\, d\theta$

we have, $y = 0$ to $\sqrt{a^2-x^2}$

$y = \sqrt{a^2-x^2}$

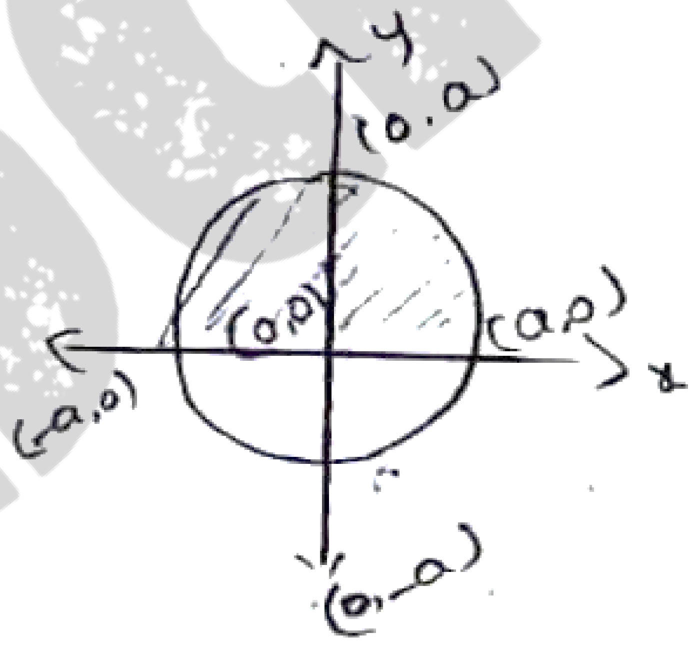
$y^2 = a^2 - x^2 \Rightarrow x^2+y^2=a^2$

This is circle with centre $(0,0)$ and $r=a$

$[(x-a^2)+(y-b^2) = r^2$, $c=(a,b)$ and $r=r]$

But $x^2+y^2=r^2 \Rightarrow \therefore a^2=r^2$

$\Rightarrow \boxed{a=r}$



$r \rightarrow 0$ to a

$\theta \rightarrow 0$ to π

$I = \int_{\theta=0}^{\pi}\int_{r=0}^{a} \sqrt{r^2}\, r\, dr\, d\theta$

$I = \int_0^{\pi} \left[\frac{r^3}{3}\right]_0^a d\theta$

$I = \int_0^{\pi} \frac{a^3}{3}\, d\theta$

$I = \frac{a^3}{3}\int_0^{\pi} d\theta$

$I = \frac{a^3}{3}\, \theta \Big|_0^{\pi}$

$\boxed{I = \frac{\pi a^3}{3}}$

3. Evaluate $\int_{y=0}^{a}\int_{x=0}^{\sqrt{a^2-y^2}} y\sqrt{x^2+y^2}\,dx\,dy$ by changing into polars.

→ In polars, $x = r\cos\theta$, $y = r\sin\theta$, $x^2+y^2 = r^2$

$$dx\,dy = r\,dr\,d\theta$$

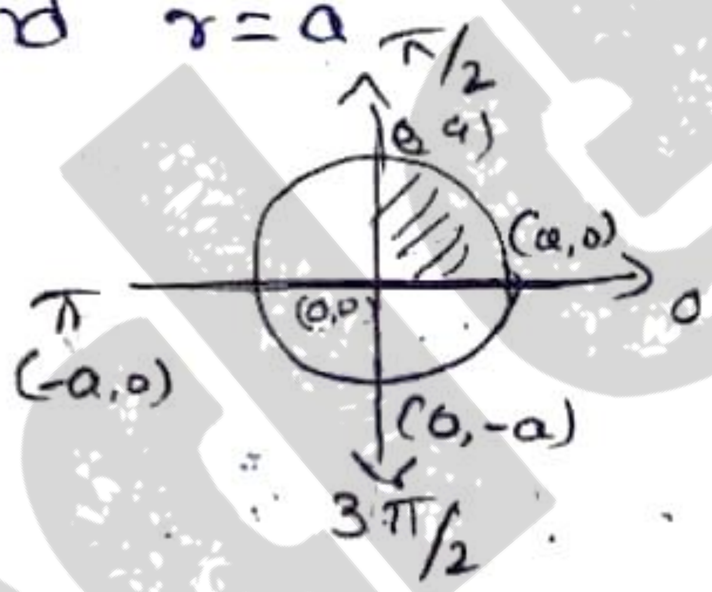
we have, $x = \sqrt{a^2-y^2}$

$x^2 = a^2 - y^2 \Rightarrow x^2+y^2 = a^2$

This is circle with centre $(0,0)$ and $r = a$

But $x^2+y^2 = r^2 \Rightarrow \therefore a^2 = r$

$\Rightarrow \boxed{a = r}$



$r \to 0$ to a

$\theta \to 0$ to $\pi/2$

$$I = \int_{\theta=0}^{\pi/2}\int_{r=0}^{a} r\sin\theta\,(\sqrt{r^2})\,r\,dr\,d\theta$$

$$I = \int_{0}^{\pi/2}\int_{0}^{a} r^3\sin\theta\,dr\,d\theta$$

$$I = \int_{0}^{\pi/2}\left[\frac{r^4}{4}\sin\theta\right]_0^a d\theta$$

$$I = \frac{1}{4}\int_{0}^{\pi/2}\left[a^4\sin\theta\right]d\theta$$

$$I = \frac{a^4}{4}\left[-\cos\theta\right]_0^{\pi/2}$$

$$I = \frac{-a^4}{4}\left[0-1\right]$$

$$\boxed{I = \frac{a^4}{4}}$$

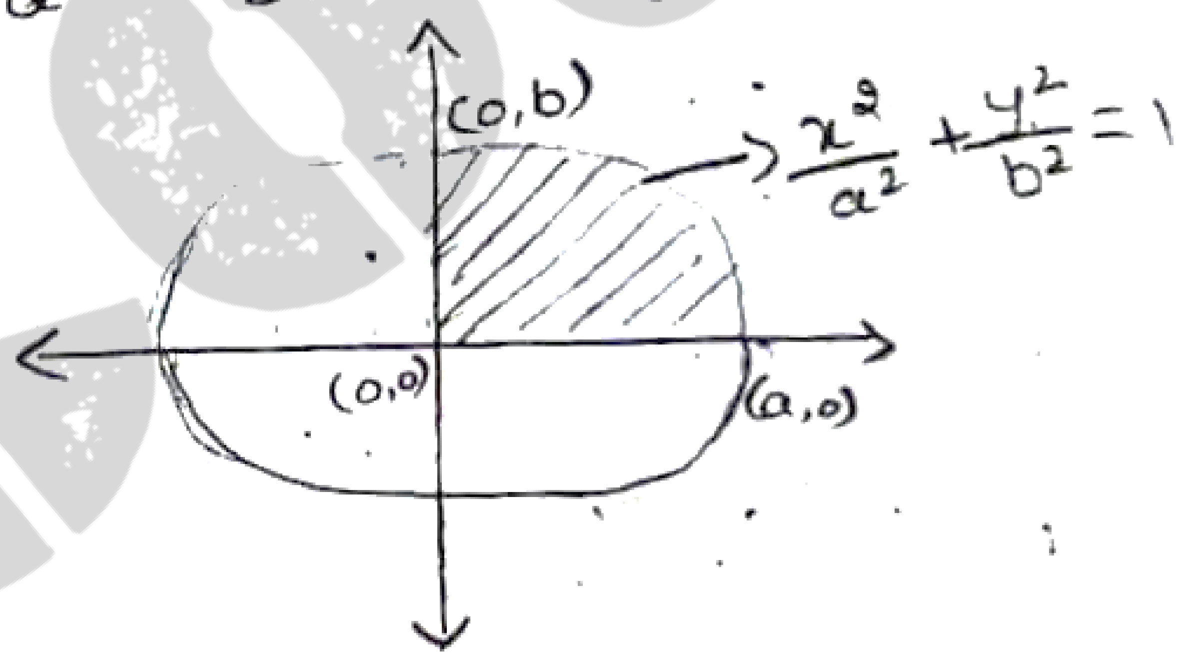
Applications to find Area and Volume

Formula

1. $\iint_R dx\,dy$ = Area of the region R in the cartesian form.
2. $\iint_R r\,dr\,d\theta$ = Area of the region R in the polar form.
3. $\iiint_V dx\,dy\,dz$ = Volume of the solid in the cartesian form.
4. $\iint_A 2\pi r^2 \sin\theta\,dr\,d\theta$ = Volume of a solid obtained by the revolution of a curve enclosing an area A about the initial line in the polar form.

June 19 *

1. Find the area of ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ by double integration.



→ Area (A) $= \iint_R dx\,dy$

Here, $A = 4A_1$

where A_1 is area in first quadrant

$\therefore A = 4A_1 = 4\int_{x=0}^{a}\int_{y=0}^{\frac{b}{a}\sqrt{a^2-x^2}} dy\,dx$

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

$y^2 = b^2\left(1-\frac{x^2}{a^2}\right)$

$y = b\sqrt{\frac{a^2-x^2}{a^2}}$

$= 4\int_{x=0}^{a} \left[y\right]_0^{\frac{b}{a}\sqrt{a^2-x^2}} dx$

$= \frac{4b}{a}\int_{x=0}^{a}\left(\sqrt{a^2-x^2}\right)dx$

$\int\sqrt{a^2-x^2}\,dx = \frac{x\sqrt{a^2-x^2}}{2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}$

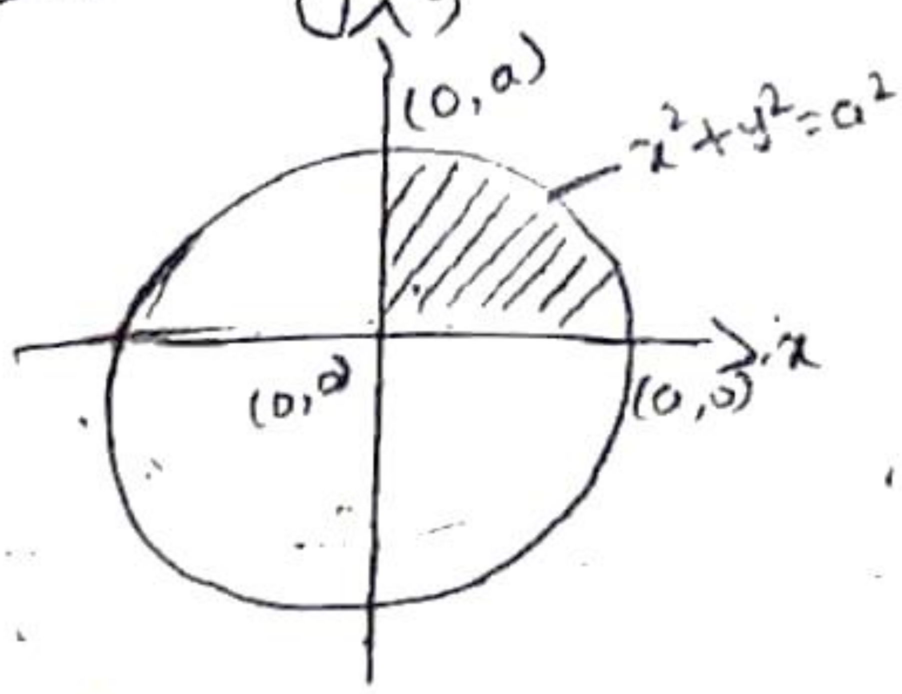
$= \frac{4b}{a}\left[\frac{x\sqrt{a^2-x^2}}{2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^a$

$= \frac{4b}{a}\left[0+\frac{a^2}{2}\left(\sin^{-1}1-\sin^{-1}0\right)\right]$

$= \frac{4ba^2}{2a}\left(\frac{\pi}{2}-0\right)$

$A = \pi ab$ sq. units

2. Find the area of circle $x^2+y^2=a^2$ by double Integration.



$\rightarrow$ $A = \iint_R dx\, dy$

Area $= A = 4A_1$

where A_1 is the Area in first quadrant.

$\therefore A = 4A_1 = 4\int_{x=0}^{a}\int_{y=0}^{\sqrt{a^2-x^2}} dy\, dx$

$A = 4\int_{x=0}^{a} [y]_0^{\sqrt{a^2-x^2}} dx$

$A = 4\int_{x=0}^{a} \sqrt{a^2-x^2}\, dx$

$A = 4\left[\frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^a$

$A = 4\left[0 + \frac{a^2}{2}(\sin^{-1}1 - \sin^{-1}0)\right]$

$A = \cancel{4}\left(\frac{a^2}{\cancel{2}}\left(\frac{\pi}{\cancel{2}} - 0\right)\right)$

$\boxed{A = \pi a^2 \text{ sq. units}}$

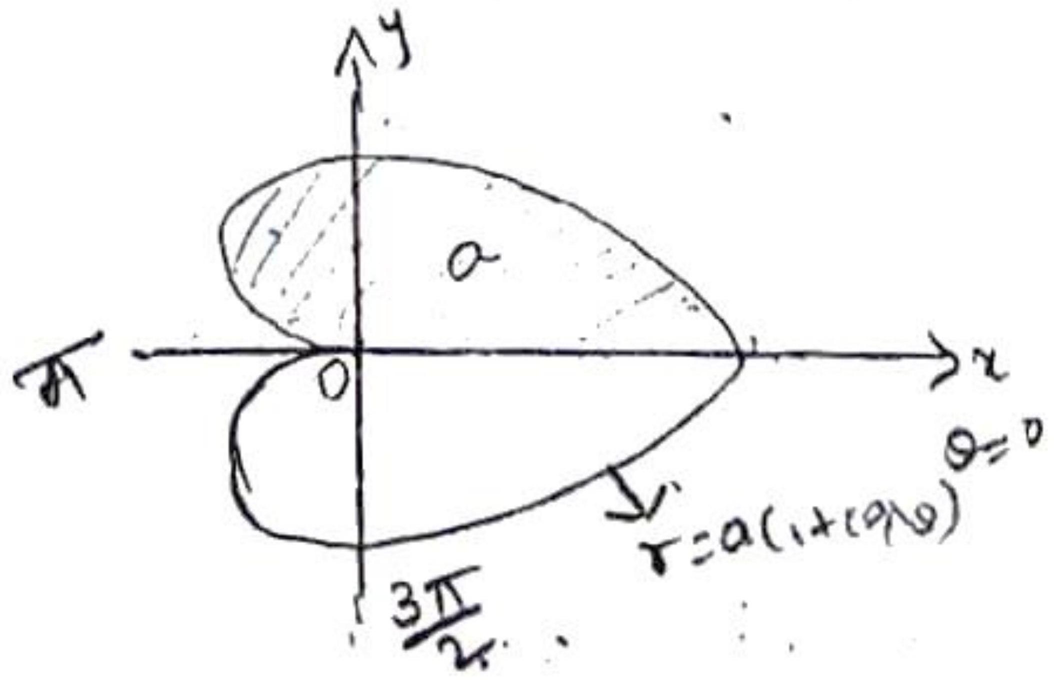
** June18, Dec19

3. Find by double integration the area enclosed by the curve $r = a(1+\cos\theta)$ b/w $\theta = 0$ to π

$\rightarrow$ $r = a(1+\cos\theta)$

It is cardiod, $\theta = 0$ to π

$r \rightarrow 0$ to $a(1+\cos\theta)$



$A = \text{Area} = \iint_R r\, dr\, d\theta$

$A = \int_{\theta=0}^{\pi}\int_{r=0}^{a(1+\cos\theta)} r\, dr\, d\theta$

$A = \int_0^{\pi} \frac{r^2}{2}\Big|_0^{a(1+\cos\theta)} d\theta$

$$A = \frac{1}{2}\int_0^{\pi} a^2\left(2\cos^2\frac{\theta}{2}\right)^2 d\theta$$

$$A = \frac{\cancel{4}a^2}{\cancel{2}}\int_0^{\pi}\cos^4\frac{\theta}{2}\, d\theta$$

put $\frac{\theta}{2} = t \Rightarrow$ $\quad \theta = 0 \quad t = 0$

$\frac{d\theta}{2} = dt \qquad \theta = \pi \quad t = \frac{\pi}{2}$

$$A = 2a^2\int_0^{\pi/2}\cos^4 t\,(2\,dt)$$

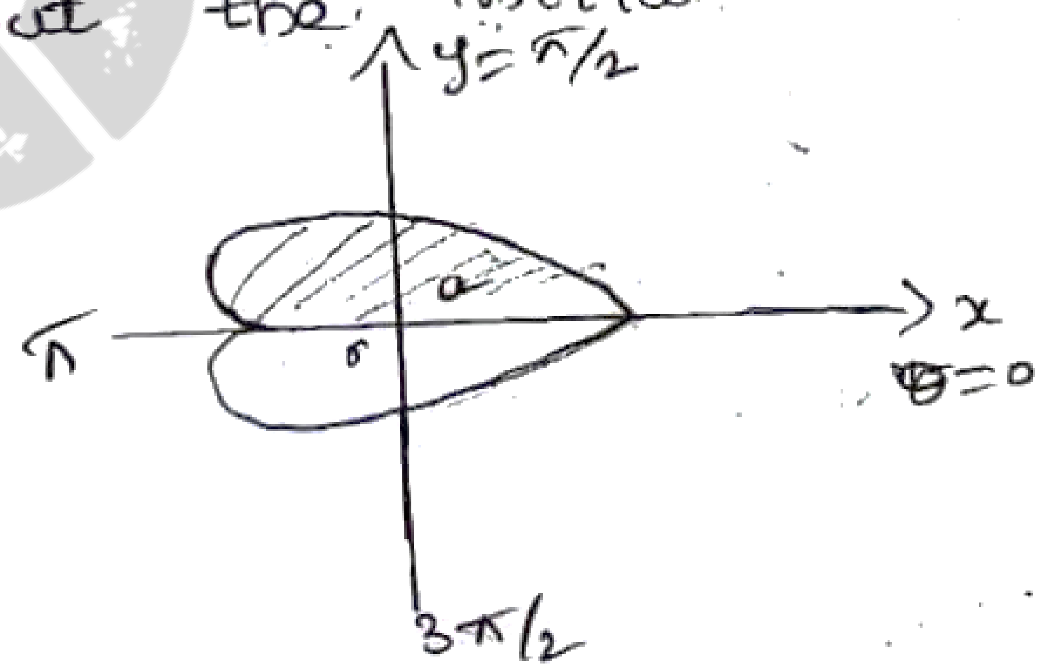
$$A = 4a^2\int_0^{\pi/2}\cos^4 t\, dt$$

(Applying Reduction formula $\int_0^{\pi/2}\cos^4 t\, dt$, $n = 4$, $k = \frac{\pi}{2}$

$$A = \cancel{4}a^2\left[\frac{3}{\cancel{4}}\times\frac{1}{2}\times\frac{\pi}{2}\right]$$

$$\boxed{A = \frac{3\pi a^2}{4} \text{ Sq. units}}$$

4. Find the volume generated by the revolution of the cardiod $r = a(1+\cos\theta)$ about the initial line.



→ $r = a(1+\cos\theta)$

$\theta = 0$ to π

$r = 0$ to $a(1+\cos\theta)$

$$\text{Volume} = V = \iint_A 2\pi r^2 \sin\theta\, dr\, d\theta$$

$$= \int_{\theta=0}^{\pi}\int_{r=0}^{a(1+\cos\theta)} 2\pi r^2 \sin\theta\, dr\, d\theta$$

$$= 2\pi\int_0^{\pi}\left[\frac{r^3}{3}\sin\theta\right]_0^{a(1+\cos\theta)} d\theta$$

$$= \frac{2\pi}{3}\int_0^{\pi}\sin\theta\left(a^3(1+\cos\theta)^3\right) d\theta$$

$$= \frac{2\pi a^3}{3}\int_0^{\pi}\sin\theta\,(1+\cos\theta)^3 d\theta$$

Put $1+\cos\theta = t \qquad \theta = 0 \quad t = 2$

$-\sin\theta\, d\theta = dt \qquad \theta = \pi \quad t = 0$

$$= \frac{2\pi a^3}{3} \int_{t=2}^{0} -dt\, t^3$$

$$= \frac{2\pi a^3}{3} \int_{t=0}^{2} t^3 dt$$

$$= \frac{\cancel{2}\pi a^3}{3} \left.\frac{t^4}{\cancel{4}}\right]_0^2$$

$$= \frac{\pi a^3}{\cancel{6}_3} [\cancel{16}^{8} - 0]$$

$$\boxed{V = \frac{8\pi a^3}{3} \text{ cubic units}}$$

5. A pyramid is bounded by 3 coordinate planes and the plane $x+2y+3z=6$. compute the volume by double integration.

$\rightarrow$ $V = \iint_A z \, dx\, dy$

Given, $x+2y+3z=6$

$$\frac{x}{6} + \frac{y}{3} + \frac{z}{2} = 1$$

$$z = 2\left(1 - \frac{x}{6} - \frac{y}{3}\right)$$

At $z=0$, $\frac{x}{6} + \frac{y}{3} = 1$

$$\Rightarrow y = 3\left(1 - \frac{x}{6}\right)$$

At $y=0$, $z=0$. $\frac{x}{6} = 1 \Rightarrow \boxed{x=6}$

$$V = \int_{x=0}^{6} \int_{y=0}^{3(1-x/6)} 2\left(1 - \frac{x}{6} - \frac{y}{3}\right) dy\, dx$$

$$V = 2\int_{x=0}^{6} \left[y - \frac{x}{6}y - \frac{y^2}{6}\right]_0^{3(1-\frac{x}{6})} dx$$

$$V = 2\int_{x=0}^{6}\left(3-\cancel{\frac{x}{2}}-\frac{x}{2}+\cancel{\frac{x^2}{12}}-\frac{3}{2}\ -\frac{x^2}{24}+\cancel{\frac{x}{2}}\right)dx$$

$$V = 2\int_0^6\left(\frac{x^2}{24}-\frac{x}{2}+\frac{3}{2}\right)dx$$

$$V = 2\left[\frac{x^3}{72}-\frac{x^2}{4}+\frac{3x}{2}\right]_0^6$$

$$V = 2\left[\frac{216}{72}-\frac{36}{4}+\frac{18}{2}\right]$$

$$V = 2\left[\frac{216-648+648}{72}\right]$$

$$V = 2(3)$$

$$\boxed{V = 6 \text{ cubic units}}$$

6. Find the volume of solid bounded by the planes $x=0$, $y=0$, $z=0$ and $x+y+z=1$

$\rightarrow$ $V = \iint_A z\ dx\ dy$

Given; $x+y+z=1 \Rightarrow z = 1-x-y$

$z=0,\ x+y=1$ $\qquad$ $z=0,\ y=0$

$\boxed{y=1-x}$ $\qquad$ $\boxed{x=1}$

$$V = \int_{x=0}^{1}\int_{y=0}^{1-x}(1-x-y)\,dy\,dx$$

$$V = \int_0^1\left[y-xy-\frac{y^2}{2}\right]_0^{1-x}dx$$

$$V = \int_0^1\left[1-x-x(1-x)-\frac{1}{2}(1-x)^2\right]dx$$

$(1-x)^2 = 1+x^2-2x$

$$V = \int_0^1\left[1-x-x+x^2-\frac{1}{2}(1+x^2-2x)\right]dx$$

$$V = \int_0^1\left(1-2x+x^2-\frac{1}{2}-\frac{x^2}{2}+x\right)dx$$

$$V = \int_0^1\left(\frac{x^2}{2}-x+\frac{1}{2}\right)dx$$

$$=\left[\frac{x^3}{6}-\frac{x^2}{2}+\frac{1}{2}x\right]_0^1$$

$$=\left[\frac{1}{6}-\cancel{\frac{1}{2}}+\cancel{\frac{1}{2}}\right]$$

$$\boxed{V=\frac{1}{6}\text{ cubic units}}$$

Jan 2018

7. Find the volume of tetrahedron bounded by the planes $x=0$, $y=0$, $z=0$ and $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$

$\rightarrow$ $$V=\iint_A z\,dx\,dy$$

Given; $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1 \Rightarrow \boxed{z=c\left(1-\frac{x}{a}-\frac{y}{b}\right)}$

At $z=0$, $\frac{x}{a}+\frac{y}{b}=1 \Rightarrow \boxed{y=b\left(1-\frac{x}{a}\right)}$

At $z=0$, $y=0$, $\frac{x}{a}=1 \Rightarrow \boxed{x=a}$

$$V=\int_{x=0}^{a}\int_{y=0}^{b(1-\frac{x}{a})} c\left(1-\frac{x}{a}-\frac{y}{b}\right)dy\,dx$$

$$V=c\int_{x=0}^{a}\left[y-\frac{x}{a}y-\frac{1}{b}\frac{y^2}{2}\right]_0^{b(1-\frac{x}{a})}dx$$

$$b^2\left(1-\frac{x}{a}\right)^2$$
$$b^2\left(1+\frac{x^2}{a^2}-\frac{2x}{a}\right)$$
$$b^2+\frac{b^2x^2}{a^2}-\frac{2xb^2}{a}$$

$$V=c\int_0^a\left[b\left(1-\frac{x}{a}\right)-\frac{x}{a}\left(b\left(1-\frac{x}{a}\right)\right)-\frac{1}{2b}\left(b^2\left(1-\frac{x}{a}\right)^2\right)\right)dx$$

$$V=c\int_0^a\left[b-\frac{bx}{a}-\cancel{\frac{xb}{a}}+\frac{bx^2}{a^2}-\frac{b}{2}-\frac{bx^2}{2a^2}+\cancel{\frac{bx}{a}}\right)dx$$

$$V=c\int_{x=0}^{a}\left[\frac{bx^2}{2a^2}-\frac{bx}{a}+\frac{b}{2}\right]dx$$

$$V=c\left[\frac{bx^3}{6a^2}-\frac{bx^2}{2a}+\frac{bx}{2}\right]_0^a$$

$$V=c\left[\frac{ba^3}{6a^2}-\cancel{\frac{ba^2}{2a}}+\cancel{\frac{ba}{2}}\right]$$

$$V=\frac{abc}{6}\text{ cubic units}$$

Beta and Gamma functions

Definitions

Beta of m, n $[\beta(m,n)] = \int_{x=0}^{x=1} x^{m-1}(1-x)^{n-1}\,dx \quad (m,n)>0 \longrightarrow (1)$

is called Beta function.

$\Gamma(n) = \int_{x=0}^{x=\infty} e^{-x} x^{n-1}\,dx \; (n>0) \longrightarrow (2)$ is called

Gamma function.

In eqn(1) put $x = \sin^2\theta$

$dx = 2\sin\theta . \cos\theta \, d\theta$

when $x=0$	when $x=1$
$0 = \sin^2\theta$	$1 = \sin^2\theta$
$\theta = \sin^{-1}(0)$	$1 = \sin\theta$
$\theta = 0$	$\theta = \sin^{-1}(1)$
	$\theta = \pi/2$

$$\therefore \beta(m,n) = \int_{\theta=0}^{\pi/2} (\sin^2\theta)^{m-1}(1-\sin^2\theta)^{n-1}\, 2\sin\theta . \cos\theta \, d\theta$$

$$\beta(m,n) = \int_0^{\pi/2} \sin^{2m-2}\theta \, \cos^{2n-2}\theta \;\; 2\sin\theta . \cos\theta . d\theta$$

$$\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \, \cos^{2n-1}\theta \, d\theta \longrightarrow (3)$$

Alternative formulae Eqn(3) & (4)

In eqn (2) put $x = t^2$

$dx = 2t\,dt$

$x=0$	$t=0$	
$x=\infty$	$t^2=\infty$	$t=\infty$

$$\therefore \sqrt{e}(m,n) = 2\int_0^{\infty} e^{-t^2} t^{2n-2} t \, dt$$

$$\Gamma(m,n) = 2\int_0^{\infty} e^{-t^2} t^{2n-1} \, dt \longrightarrow (4)$$

Properties of Beta and Gamma functions

1. $\beta(m,n) = \beta(n,m)$

Proof:- LHS $= \beta(m,n) = \int_{x=0}^{x=1} x^{m-1}(1-x)^{n-1}\,dx$

Put $1-x = y \Rightarrow x = 1-y$ $\quad x=0 \quad y=1$

$dx = -dy$ $\quad \boxed{x=1} \quad \boxed{y=0}$

$$\beta(m,n) = \int_{y=1}^{0} (1-y)^{m-1} y^{n-1}(-dy)$$

$$\beta(m,n) = \int_{0}^{1} (1-y)^{m-1} y^{n-1}\,dy = \beta(n,m) = \text{RHS}$$

$$\boxed{\beta(m,n) = \beta(n,m)}$$

2. (a) P.T $\Gamma(n+1) = n\Gamma n$

(b) $\Gamma n+1 = n!$ where n is a positive integers.

Proof:- (a) we have, $\Gamma(n) = \int_{x=0}^{\infty} e^{-x} x^{n-1}\,dx$

Replace n by $n+1$

$$\Rightarrow \Gamma(n+1) = \int_{x=0}^{\infty} \underset{\text{II}}{e^{-x}}\, \underset{\text{I}}{x^{n}}\,dx$$

on Integrating by parts on RHS,

$\text{I}\int\text{II} - \int\int \text{II}\, d(\text{I})$

$$\Rightarrow x^n \int_0^{\infty} e^{-x}dx - \int_0^{\infty}\int_0^{\infty} e^{-x}dx\, \frac{d}{dx}(x^n)$$

$$\Rightarrow x^n \frac{e^{-x}}{-1}\Big|_0^{\infty} - \int_0^{\infty} \frac{e^{-x}}{-1}(nx^{n-1})\,dx$$

$$\Rightarrow -(0-0) + \int_0^{\infty} e^{-x}(n x^{n-1})\, dx$$

$$\Rightarrow \Gamma(n+1) = n \int_0^{\infty} e^{-x} x^{n-1}\, dx$$

$$\Gamma(n+1) = n\,\Gamma n$$

b. $\Gamma(n+1) = n!$

we have $\Gamma(n+1) = n\,\Gamma(n)$

Replace n by $n-1$

$$\Gamma(n) = (n-1)\,\Gamma(n-1)$$

Replace n by $(n-1) \Rightarrow \Gamma(n-1) = n-2\,\Gamma(n-2)$

Replace n by $(n-2) \Rightarrow \Gamma(n-2) = n-3\,\Gamma(n-3)$

$$\vdots$$

$$\Gamma(3) = 2\cdot\Gamma(2)$$

$$\Gamma(2) = 1\,\Gamma(1)$$

Now, By back substitution

$$\Gamma(n+1) = n(n-1)(n-2)(n-3) \cdots 3, 2, 1, \Gamma(1)$$

$$\Gamma(n+1) = n!\,\Gamma 1$$

By definition $\Gamma(n) = \int_{x=0}^{\infty} e^{-x} x^{n-1}\, dx$

put $n=1$, $\Gamma(1) = \int_0^{\infty} e^{-x} x^{1-1}\, dx$

$$= \left.\frac{e^{-x}}{-1}\right|_0^{\infty}$$

$$= -(e^{-\infty} - e^{0})$$

$$= -(0-1) = 1 = \Gamma(1)$$

$\therefore \Gamma(n+1) = n!\,(1)$

$$\Gamma(n+1) = n!$$

MEX V.Imp 100%

Relationship b/w β and Γ function:-

$$\beta(m,n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$$

Proof:- $\beta(m,n) = 2\int_{\theta=0}^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta \longrightarrow (1)$

$\Gamma(n) = 2\int_{x=0}^{\infty} e^{-x^2} x^{2n-1} dx \longrightarrow (2)$

$\Gamma(m) = 2\int_{y=0}^{\infty} e^{-y^2} y^{2m-1} dy \longrightarrow (3)$

$\Gamma(m+n) = 2\int_{r=0}^{\infty} e^{-r^2} r^{2(m+n)-1} dr \longrightarrow (4)$

Now, $\Gamma(m)\,\Gamma(n) = 4\int_{x=0}^{\infty}\int_{y=0}^{\infty} e^{-(x^2+y^2)} x^{2n-1} y^{2m-1} dx\,dy \longrightarrow (5)$

Put $x = r\cos\theta$ $\quad y = r\sin\theta$

$x^2+y^2 = r^2(\cos^2\theta + \sin^2\theta)$

$= r^2(1)$

$x^2+y^2 = r^2$

$dx\,dy = r\,dr\,d\theta$

$r \to 0$ to ∞

$\theta \to 0$ to $\pi/2$

(5) $\Rightarrow$ $\Gamma(m)\,\Gamma(n) = 4\int_{r=0}^{\infty}\int_{\theta=0}^{\pi/2} e^{-r^2}(r\cos\theta)^{2n-1}(r\sin\theta)^{2m-1} r\,dr\,d\theta$

$= 4\int_{r=0}^{\infty}\int_{\theta=0}^{\pi/2} e^{-r^2} r^{2m+2n-1} \cos^{2n-1}\theta \sin^{2m-1}\theta \, dr\,d\theta$

$= 2\int_{r=0}^{\infty} e^{-r^2} r^{2(m+n)-1} dr . \, 2\int_{\theta=0}^{\pi/2} \cos^{2n-1}\theta \sin^{2m-1}\theta \, d\theta$

from (1) and (4)

$$\Gamma(m)\ \Gamma(n) = \Gamma(m+n)\ \beta(m,n)$$

$$\therefore \boxed{\beta(m,n) = \frac{\Gamma(m)\ \Gamma(n)}{\Gamma(m+n)}}$$

TO S.T $\Gamma(1/2) = \sqrt{\pi}$

~~→~~ Using relationship b/w β and Γ function.

→ we have $\beta(m,n) = \dfrac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$

$$\beta(1/2, 1/2) = \frac{\Gamma(1/2)\cdot\Gamma(1/2)}{\Gamma(1/2+1/2)}$$

$$= \frac{(\Gamma 1/2)^2}{\Gamma 1}$$

$$\beta(1/2, 1/2) = (\Gamma 1/2)^2 \longrightarrow (1)$$

consider $\beta(m,n) = 2\int_{\theta=0}^{\pi} \sin^{2m-1}\theta \cos^{2n-1}\theta\, d\theta \longrightarrow (1)$

put $m=n=1/2$

$$\beta(1/2, 1/2) = 2\int_{\theta=0}^{\pi} \sin^0\theta \cos^0\theta\, d\theta$$

$$= 2\,\theta \Big|_0^{\pi/2}$$

$$= 2(\pi/2 - 0)$$

$$\beta(1/2, 1/2) = \pi \longrightarrow (2)$$

Eqn (1) & (2) $\Rightarrow$ $\Gamma(1/2)^2 = \pi$

$$\boxed{\Gamma(1/2) = \sqrt{\pi}}$$

Note:-

1. $\Gamma(n) = (n-1)\Gamma(n-1)$, $\Gamma(n) = (n-1)!$ if n is a positive integer.
2. $\beta(m,n) = \dfrac{\Gamma(m)\cdot\Gamma(n)}{\Gamma(m+n)}$ (3) $\Gamma 1 = 1$, $\Gamma(1/2) = \sqrt{\pi}$, $\Gamma(1/4)\Gamma(3/4) = \pi\sqrt{2}$

b) P.T $\Gamma \tfrac{1}{2} = \sqrt{\pi}$ using definitions of $\Gamma(n)$

$$\Gamma(n) = 2 \int_{x=0}^{\infty} e^{-x^2} x^{2n-1} \, dx, \quad n = \tfrac{1}{2}$$

$$\Gamma(\tfrac{1}{2}) = 2 \int_{x=0}^{\infty} e^{-x^2} \, dx \longrightarrow (1)$$

$$\Gamma(m) = 2 \int_{y=0}^{\infty} e^{-y^2} y^{2m-1} \, dy \quad m = \tfrac{1}{2}$$

$$\Gamma(\tfrac{1}{2}) = 2 \int_{y=0}^{\infty} e^{-y^2} \, dy \longrightarrow (2)$$

from (1) & (2)

$$\left[\Gamma(\tfrac{1}{2})\right]^2 = 4 \int_{x=0}^{\infty} \int_{y=0}^{\infty} e^{-(x^2+y^2)} \, dx \cdot dy$$

put $x = r\cos\theta$ $\quad y = r\sin\theta$

$dx\,dy = r\,dr\,d\theta$

$r \to 0$ to ∞

$\theta \to 0$ to $\pi/2$

$x^2 + y^2 = r^2$

$$\left[\Gamma(\tfrac{1}{2})\right]^2 = 4 \int_{r=0}^{\infty} \int_{\theta=0}^{\pi/2} e^{-r^2} r \, dr \, d\theta$$

put $r^2 = t$

$2r\,dr = dt$ $\quad r = 0 \quad t = 0$

$\quad r = \infty \quad t = \infty$

$$= 4 \int_{\theta=0}^{\pi/2} \int_{t=0}^{\infty} e^{-t} \frac{dt}{2} \, d\theta$$

$$= 2 \int_{\theta=0}^{\pi/2} \left. \frac{e^{-t}}{-1} \right|_0^{\infty} d\theta$$

$$= 2 \int_0^{\pi/2} -(e^{-\infty} - e^{0}) \, d\theta$$

$$= 2 \int_0^{\pi/2} d\theta$$

$$= 2 \left. \theta \right|_0^{\pi/2} = 2\left[\tfrac{\pi}{2} - 0\right] = \pi$$

$$\boxed{\Gamma \tfrac{1}{2} = \sqrt{\pi}}$$

Problems

1. S.T $\beta(m+1, n) + \beta(m, n+1) = \beta(m, n)$

$\rightarrow$ $LHS = \frac{\Gamma(m+1)\Gamma(n)}{\Gamma(m+n+1)} + \frac{\Gamma(m)\Gamma(n+1)}{\Gamma(m+n+1)}$

$= \frac{m\Gamma(m)\Gamma(n)}{(m+n)\Gamma(m+n)} + \frac{\Gamma(m)\, n\Gamma(n)}{(m+n)\Gamma(m+n)}$

$= \frac{\Gamma(m)\Gamma(n)\cdot\cancel{(m+n)}}{\cancel{(m+n)}\Gamma(m+n)}$

$= \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} = \beta(m,n) = RHS$

2. Evaluate (i) $\frac{\Gamma(3)\Gamma(2.5)}{\Gamma(5.5)}$, (ii) $\frac{6\Gamma 8/3}{5\Gamma 2/3}$, (iii) $\Gamma(-7/2)$ (iv) $\beta(7/2, -1/2)$

$\rightarrow$ we have, $\Gamma(n+1) = n\Gamma(n) \rightarrow$ (+)ve ~~integer~~ fraction

$\Gamma(n) = \frac{\Gamma n+1}{n} \rightarrow$ (-)ve negative fraction

$\Gamma n+1 = n!$

n by $(n-1) \Rightarrow \Gamma n = (n-1)! \rightarrow$ (+)ve Integer

(i) $\Gamma 3 = (3-1)! = 2! = 2$

$\Gamma 2.5 = \Gamma(1.5+1) = 1.5\,\Gamma 1.5$

$\Gamma 1.5 = \Gamma(0.5+1) = 0.5\,\Gamma 0.5$

$\Gamma 5.5 = \Gamma(4.5+1) = 4.5\,\Gamma 4.5$

$\Gamma 4.5 = \Gamma(3.5+1) = 3.5\,\Gamma 3.5$

$\Gamma 3.5 = \Gamma(2.5+1) = 2.5\,\Gamma 2.5$

$\Gamma 2.5 = \Gamma(1.5+1) = 1.5\,\Gamma 1.5$

$\Gamma 1.5 = \Gamma(0.5+1) = 0.5\,\Gamma 0.5$

$\Rightarrow \frac{\Gamma(3)\Gamma 2.5}{\Gamma 5.5} = \frac{\cancel{2\times 1.5\times 0.5\,\Gamma 0.5}}{4.5\times 3.5\times 2.5\,\cancel{\times 1.5\times 0.5\,\Gamma 0.5}} = \frac{2}{9/2\times 7/2\times 5/2} = \frac{16}{315}$

b) $\Gamma 8/3 = \Gamma(5/3+1) = 5/3\ \Gamma 5/3 = \Gamma(2/3+1) = 2/3\ \Gamma 2/3$

$\Gamma 5/3 = \Gamma(2/3+1) = 2/3\ \Gamma 2/3$

$$\frac{6\,\Gamma 8/3}{5\,\Gamma 2/3} = \frac{\cancel{6}^{2} \times 5/3 \times \cancel{2/3\,\Gamma 2/3}}{5 \times \cancel{2/3}\,\cancel{\Gamma 2/3}} = \frac{\cancel{10}^{2} \times \cancel{2/3}}{\cancel{3}} = \cancel{\frac{4}{3}}\quad 2$$

c) $\Gamma -7/2$

$$\Gamma -7/2 = \frac{\Gamma -7/2+1}{-7/2} = \frac{\Gamma -5/2}{-7/2} = \frac{-2}{7}\,\Gamma -5/2$$

$$\Gamma -5/2 = \frac{\Gamma -5/2+1}{-5/2} = \frac{\Gamma -3/2}{-5/2} = \frac{-2}{7} \times \frac{-2}{5}\,\Gamma -3/2$$

$$\Gamma -3/2 = \frac{\Gamma -3/2+1}{-3/2} = \frac{\Gamma -1/2}{-3/2} = \frac{-2}{7} \times \frac{-2}{5} \times \frac{-2}{3}\,\Gamma -1/2$$

$$\Gamma -1/2 = \frac{\Gamma -1/2+1}{-1/2} = \frac{\Gamma 1/2}{-1/2} = \frac{-2}{7} \times \frac{-2}{5} \times \frac{-2}{3} \times \frac{-2}{1} \times \Gamma \frac{1}{2}$$

$$= \frac{16\sqrt{\pi}}{105} = RHS$$

d) $\beta(\overset{m}{7/2}, \overset{n}{-1/2})$

$$\frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)} = \beta(m,n)$$

$$\beta(7/2, -1/2) = \frac{\Gamma(7/2)\,\Gamma(-1/2)}{\Gamma(7/2 + (-1/2))} = \frac{\Gamma(7/2)\,\Gamma(-1/2)}{\Gamma 3}$$

$\Gamma 7/2 = \Gamma(\frac{5}{2}+1) = \frac{5}{2}\,\Gamma(5/2)$

$\Gamma 5/2 = \Gamma(\frac{3}{2}+1) = \frac{3}{2}\,\Gamma(3/2)$

$\Gamma 3/2 = \Gamma(1/2+1) = \frac{1}{2}\,\Gamma(1/2)$

$\Gamma 3 = (3-1)!$
$= 2! = 2$

$$\Gamma(-1/2) = \frac{\Gamma -1/2+1}{-1/2} = \frac{-2}{1}\,\Gamma 1/2$$

$$\Rightarrow \beta(7/2, -1/2) = \frac{\frac{5}{2} \times \frac{3}{2} \times \frac{1}{2} \times -2 \times \Gamma \frac{1}{2}}{2}$$

$$= \frac{\cancel{5} \times 3 \times 1 \times -\cancel{2}}{\cancel{2} \times 2 \times 2 \times 2} (\pi)$$

$$= \frac{-15\pi}{8}$$

3) show that $\int_0^1 [\log(1/y)]^{n-1} dy = \Gamma(n)$

$$LHS = \int_0^1 [\log(1/y)]^{n-1} dy$$

put $\log(1/y) = t$

$$1/y = e^t$$

$$y = e^{-t}$$

$$dy = -e^{-t} dt$$

$y = 0$ $\quad 0 = e^{-t}$

$\log 0 = -t$

$-\infty = -t$

$t = \infty$

$y = 1$ $\quad 1 = e^{-t}$

$\log 1 = -t$

$t = 0$

$$\int_{t=\infty}^{0} t^{n-1} - e^{-t} dt$$

$$= \int_{t=0}^{\infty} t^{n-1} e^{-t} dt$$

$$= \int_{t=0}^{\infty} e^{-t} t^{n-1} dt$$

$$= \Gamma(n)$$

Evaluation of definite integrals by converting into gamma functions

Step-1: By the definitions of Gamma function in two of the standard form

$$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1}\, dx;$$

$$\Gamma(n) = 2\int_0^\infty e^{-x^2} x^{2n-1}\, dx.$$

Step 2: correlate the value corresponding to $(n-1)$ (or) $(2n-1)$ to find n.

1. Evaluate $\int_0^\infty x^{3/2} e^{-x}\, dx$

→ W.K.T $\Gamma(x) = \int_{x=0}^\infty e^{-x} x^{n-1}\, dx \longrightarrow (1)$

$$\int_0^\infty x^{3/2} e^{-x}\, dx$$

$$\frac{3}{2} = n-1$$

$$n = \frac{3}{2} + 1 = \frac{5}{2}$$

$$\boxed{I = \Gamma 5/2}$$

$$\Gamma 5/2 = \Gamma 3/2 + 1 = \frac{3}{2}\Gamma 3/2$$

$$\Gamma 3/2 = \Gamma 1/2 + 1 = \frac{1}{2}\cdot\Gamma(1/2)$$

$$I = \Gamma 5/2 = 3/2 \cdot \frac{1}{2}\,\Gamma\frac{1}{2}$$

$$\boxed{I = \frac{3\sqrt{\pi}}{4}}$$

$$I = \int_0^\infty x^{1/4} e^{-\sqrt{x}}\, dx$$

$$I = \int_0^\infty x^{1/4} e^{-\sqrt{x}}\, dx$$

put $\sqrt{x} = t$ $\quad x = 0,\ t = 0$

$x = t^2$ $\quad x = \infty,\ t = \infty$

$dx = 2t\,dt$

$$I = \int_{t=0}^\infty e^{-t} (t^2)^{1/4}\, 2t\, dt \qquad 1+\tfrac{1}{2} = \tfrac{3}{2}$$

$$I = 2\int_{t=0}^\infty e^{-t} t^{3/2}\, dt$$

compare with $\Gamma(n) = \int_0^\infty e^{-x} x^{n-1}\, dx$

$$n-1 = \frac{3}{2} \Rightarrow n = \frac{3}{2}+1 \Rightarrow \boxed{n = \frac{5}{2}}$$

$$\boxed{I = \Gamma 5/2}$$

$$I = \Gamma 5/2 = 2\left(\frac{3}{2} \times \frac{1}{2}\sqrt{\pi}\right)$$

$$\boxed{I = \frac{3\sqrt{\pi}}{2}}$$

3. Show that $\int_0^\infty \sqrt{y}\, e^{-y^2} dy \times \int_0^\infty \frac{e^{-y^2}}{\sqrt{y}} dy = \frac{\pi}{2\sqrt{2}}$

$\rightarrow$ Let $I_1 = \int_0^\infty \sqrt{y}\, e^{-y^2} dy$

$$= \int_0^\infty y^{1/2} e^{-y^2} dy \longrightarrow (1)$$

$$I_2 = \int_0^\infty \frac{e^{-y^2}}{\sqrt{y}} dy$$

$$= \int_0^\infty y^{-1/2} e^{-y^2} dy \longrightarrow (2)$$

W.K.T $\Gamma(n) = 2\int_{x=0}^\infty e^{-x^2} x^{2n-1}\, dx$

$$\frac{1}{2}\Gamma(n) = \int_{x=0}^\infty e^{-x^2} x^{2n-1}\, dx \longrightarrow (3)$$

compare (1) and (3)

$\Rightarrow \quad 2n-1 = \frac{1}{2}$

$2n = \frac{1}{2}+1$

$2n = \frac{3}{2}$

$\boxed{n = \frac{3}{4}}$

compare (1) and (2)

$2n-1 = -\frac{1}{2}$

$2n = -\frac{1}{2}+1$

$2n = \frac{1}{2}$

$\boxed{n = \frac{1}{4}}$

$I_1 = \frac{1}{2}\Gamma 3/4$

$I_2 = \frac{1}{2}\Gamma \frac{1}{4}$

$\therefore I_1 \times I_2 = \frac{1}{2}\Gamma 3/4 \times \frac{1}{2}\Gamma 1/4$ $\qquad \left[\Gamma 3/4 \, \Gamma 1/4 = \pi\sqrt{2}\right]$

$= \frac{1}{4}\pi\sqrt{2}$

$= \frac{\pi\sqrt{2}}{2\times 2} = \frac{\pi\cancel{\sqrt{2}}}{2\times\cancel{\sqrt{2}}\times\sqrt{2}}$

$= \frac{\pi}{2\sqrt{2}}$

4. Show that $\int_0^\infty x\, e^{-x^8}\,dx \times \int_0^\infty x^2 e^{-x^4}\,dx = \frac{\pi}{16\sqrt{2}}$

$\rightarrow$ Let $I_1 = \int_0^\infty x\, e^{-x^8}\,dx$

put $x^8 = t \Rightarrow x = t^{1/8}$

$\Rightarrow 8x^7\,dx = dt$

$x\,dx = \frac{dt}{8x^6}$

$= \frac{dt}{8(t^{1/8})^6}$

$= \frac{dt}{8\,t^{3/4}}$

$x=0, \quad t=0$

$x=\infty, \quad t=\infty$

$I_2 = \int_0^\infty x^2 e^{-x^4}\,dx$

put $x^4 = t \Rightarrow (x = t^{1/4})$

$\Rightarrow 4x^3\,dx = dt$

$x^2 dx = \frac{dt}{4x}$

$= \frac{dt}{4(t^{1/4})}$

$x=0, \ t=0$

$x=\infty, \ t=\infty$

$$I_1 = \int_{t=0}^{\infty} e^{-t} \frac{dt}{8t^{3/4}} \qquad I_2 = \int_{t=0}^{\infty} e^{-t} \frac{dt}{4t^{1/4}}$$

$$I_1 = \frac{1}{8}\int_{t=0}^{\infty} e^{-t}\, t^{-3/4}\, dt \longrightarrow (1) \qquad I_2 = \frac{1}{4}\int_{t=0}^{\infty} e^{-t}\, t^{-1/4}\, dt \longrightarrow (2)$$

e

w.k.T $\Gamma(n) = \int_0^{\infty} e^{-x} x^{n-1} dx \longrightarrow (3)$

compare (1) and (3)

$n-1 = -3/4$

$n = -3/4 + 1$

$n = 1/4$

$\therefore I_1 = \frac{1}{8}\Gamma 1/4$

compare (2) and (3)

$n-1 = -1/4$

$n = -1/4 + 1$

$n = 3/4$

$I_2 = \frac{1}{4}\Gamma 3/4$

$$I_1 \times I_2 = \frac{1}{8}\Gamma\frac{1}{4} \times \frac{1}{4}\Gamma 3/4$$

$$= \frac{1}{32}\left[\Gamma\frac{1}{4} \times \Gamma 3/4\right] \qquad \left(\because \Gamma\frac{1}{4} \times \Gamma 3/4 = \pi\sqrt{2}\right)$$

$$= \frac{\pi\sqrt{2}}{16 \times \sqrt{2} \times \sqrt{2}} = \frac{\pi}{16 \times \sqrt{2}}$$

$$\boxed{I = \frac{\pi}{16\sqrt{2}}}$$

5. Show That $\int_0^{\infty} \frac{e^{-x^2}}{\sqrt{x}} dx \times \int_0^{\infty} e^{-x^4} x^2 dx = \frac{\pi}{4\sqrt{2}}$

$\rightarrow I_1 = \int_0^{\infty} e^{-x^2} x^{-1/2} dx$ — (1) $\qquad I_2 = \int_0^{\infty} e^{-x^4} x^2 dx$ — (2)

w.k.T $\Gamma(n) = \int_0^{\infty} e^{-x} x^{n-1} dx$ — (3)

$\frac{1}{2}\Gamma(x) = \int_0^{\infty} e^{-x} x^{2n-1} dx$ — (4)

compare (1) & (4)

~~compare (2) & (3)~~

~~$2n-1=2$~~

$2n-1 = -1/2$

$2n = -1/2 + 1$

$2n = 1/2$

$\boxed{n = 1/4}$

$$\boxed{I_1 = \frac{1}{2}\Gamma\frac{1}{4}}$$

put $x^4 = t \Rightarrow (x = t^{1/4})$

$4x^3\,dx = dt$

$x^2\,dx = \dfrac{dt}{4x}$ $\qquad$ $x=0,\ t=0$

$\qquad\qquad$ $x=\infty,\ t=\infty$

$= \dfrac{dt}{4(t)^{1/4}}$

$$I_2 = \int_{t=0}^{\infty} e^{-t}\,\frac{dt}{4\,t^{1/4}}$$

$$I_2 = \frac{1}{4}\int_{t=0}^{\infty} e^{-t}\,t^{-1/4}\,dt \longrightarrow (2)$$

compare (2) & (3)

$n-1 = -1/4$

$n = -1/4 + 1$

$\boxed{n = 3/4}$

$I_2 = \frac{1}{4}\Gamma 3/4$

$\therefore\ I_1 \times I_2 = \frac{1}{2}\Gamma\frac{1}{4} \times \frac{1}{4}\Gamma 3/4$

$= \frac{1}{8}\,\pi\sqrt{2}$

$= \dfrac{\pi\cancel{\sqrt{2}}}{4\times\sqrt{2}\times\cancel{\sqrt{2}}}$

$\boxed{I = \dfrac{\pi}{4\sqrt{2}}}$

Problems on converting an integral into beta function and Evaluation by transforming into Gamma function.

By the defn of β function, we have

$$\int_0^1 x^{m-1}(1-x)^{n-1}\,dx = \beta(m,n)$$

$$\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta = \frac{1}{2}\beta(m,n)$$

correlate with $(m-1)$ and $(n-1)$ (or) $(2m-1)$ & $(2n-1)$

to find the values of m & n

Note:- $\cancel{\int_0^{\pi/2} \sqrt{\tan\theta}\, d\theta = \frac{\pi}{\sqrt{2}}}$

1. Express the following integrals in terms of β function and hence evaluate.

(a) $I = \int_0^2 (4-x^2)^{3/2}\, dx$

→ put $x = 2\sin\theta$

$dx = 2\cos\theta \, d\theta$

$$(4-x^2)^{3/2} = (4-4\sin^2\theta)^{3/2}$$

$$= [4(1-\sin^2\theta)]^{3/2}$$

$$= 4^{3/2}(1-\sin^2\theta)^{3/2}$$

$$= (2^{\cancel{2}})^{3/\cancel{2}}(\cos^{\cancel{2}}\theta)^{3/\cancel{2}}$$

$$= 8\cos^3\theta.$$

$x=0$ $\quad 0 = 2\sin\theta$

$0 = \sin\theta$

$\boxed{\theta = 0}$

$x = 2$ $\quad 2 = 2\sin\theta$

$\sin\theta = \frac{2}{2} = 1 \Rightarrow \theta = \sin^{-1}(1)$

$\boxed{\theta = \pi/2}$

$$I = \int_0^{\pi/2} 8\cos^3\theta \; 2\cos\theta \, d\theta$$

$$I = 16\int_0^{\pi/2} \sin^0 \cos^4\theta \, d\theta$$

$2m-1=0$ $\quad$ $m=1/2$ $\qquad$ $2n-1=4$ $\quad$ $2n=4+1$ $\quad$ $n=5/2$

$$I = \cancel{16}^{8} \cdot \frac{1}{\cancel{2}} \beta(1/2, 5/2)$$

$$I = 8\left[\frac{\Gamma 1/2 \cdot \Gamma 5/2}{\Gamma(1/2+5/2)}\right] = \frac{\cancel{8}^{\cancel{4}\,2} \sqrt{\pi} \cdot 3/\cancel{2} \times 1/\cancel{2} \sqrt{\pi}}{(3-1)!} = \frac{\cancel{6}\pi}{\cancel{2}}$$

$$\boxed{I = 3\pi}$$

(b) $I = \int_0^{\infty} \frac{dx}{1+x^4}$

→ put $x^4 = \tan^2\theta \Rightarrow x = (\tan^2\theta)^{1/4}$

$x = \tan^{1/2}\theta$

$dx = \frac{1}{2}\tan^{-1/2}\theta \sec^2\theta \, d\theta$

$x=0 \quad \theta=0 \qquad x=\infty, \; \theta=\pi/2$

$$I = \int_{\theta=0}^{\pi/2} \frac{1}{(1+\tan^2\theta)} \frac{1}{2}\tan^{-1/2}\theta \sec^2\theta \, d\theta$$

$$I = \frac{1}{2}\int_0^{\pi/2} \frac{1}{\cancel{\sec^2\theta}} \tan^{-1/2}\theta \cancel{\sec^2\theta} \, d\theta$$

$$I = \frac{1}{2}\int_0^{\pi/2} \frac{\sin^{-1/2}\theta}{\cos^{-1/2}\theta} d\theta$$

$$I = \frac{1}{2}\int_0^{\pi/2} \sin^{-1/2}\theta \cos^{1/2}\theta \, d\theta$$

$2m-1 = -1/2 \qquad 2n-1 = \frac{1}{2}$

$2m = 1/2 \qquad 2n = 3/2$

$\boxed{m = 1/4} \qquad \boxed{n = 3/4}$

$$= \frac{1}{2} \times \frac{1}{2} \; \beta(1/4, 3/4)$$

$$= \frac{1}{4} \left[\frac{\Gamma 1/4 \; \Gamma 3/4}{\Gamma(1/4 + 3/4)} \right]$$

$$= \frac{1}{4} \left[\frac{\pi \sqrt{2}}{1} \right]$$

$$= \frac{\pi \cancel{\sqrt{2}}}{2 \times \sqrt{2} \times \cancel{2}}$$

$$\boxed{I = \frac{\pi}{2\sqrt{2}}}$$

v.v.Imp. * * * * * * * * * * * * *

2. Show that $\int_0^{\pi/2} \frac{d\theta}{\sqrt{\sin\theta}} \times \int_0^{\pi/2} \sqrt{\sin\theta}\, d\theta = \pi$

→ we have

$$\beta(m,n) = 2 \int_0^{\pi/2} \sin^{2m-1}\theta \; \cos^{2n-1}\theta \; d\theta$$

put $2m-1 = p$, $2n-1 = q$

$$\therefore \frac{1}{2} \beta(p,q) = \int_0^{\pi/2} \sin^p\theta . \cos^q\theta \; d\theta$$

$$\boxed{\frac{1}{2} \beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right) = \int_0^{\pi/2} \sin^p\theta . \cos^q\theta \, d\theta}$$

$$\Rightarrow I_1 = \int_0^{\pi/2} \frac{1}{\sqrt{\sin\theta}} d\theta = \int_0^{\pi/2} \sin^{-1/2}\theta . \cos^0 d\theta .$$

$p = -1/2$, $q = 0$

$$= \frac{1}{2} \beta\left(\frac{-1/2 + 1}{2}, \frac{0+1}{2}\right)$$

$$= \frac{1}{2} \beta\left(\frac{1}{4}, \frac{1}{2}\right)$$

$$I_1 = \frac{1}{2} \left(\frac{\Gamma 1/4 \; \Gamma 1/2}{\Gamma(1/4 + 1/2)}\right) = \frac{1}{2} \left[\frac{\Gamma 1/4 \, \Gamma 1/2}{\Gamma(3/4)}\right]$$

$$I_2 = \int_0^{\pi/2} \sqrt{\sin\theta}\, d\theta = \int_0^{\pi/2} \sin^{1/2}\theta \; \cos^0\theta \; d\theta$$

$$p = \tfrac{1}{2} \qquad q = 0$$

$$= \frac{1}{2}\beta\left(\frac{\frac{1}{2}+1}{2}, \frac{0+1}{2}\right)$$

$$= \frac{1}{2}\beta\left(\tfrac{3}{4}, \tfrac{1}{2}\right)$$

$$= \frac{1}{2}\left[\frac{\Gamma 3/4 \; \Gamma 1/2}{\Gamma(3/4 + 1/2)}\right]$$

$$I_2 = \frac{1}{2}\left[\frac{\Gamma(3/4)\;\Gamma(1/2)}{\Gamma(5/2)}\right]$$

$$I_1 \times I_2 = \frac{1}{2}\left[\frac{\Gamma 1/4 \times \Gamma 1/2}{\cancel{\Gamma 3/4}}\right] \times \frac{1}{2}\left[\frac{\cancel{\Gamma 3/4}\;\Gamma 1/2}{\Gamma 5/4}\right]$$

$$= \frac{1}{4}\,\frac{\Gamma 1/4 \;\pi}{\Gamma 5/4} \qquad \Gamma 5/4 = \Gamma 1/4 + 1 = \tfrac{1}{4}\Gamma \tfrac{1}{4}$$

$$= \frac{1}{\cancel{4}}\,\frac{\cancel{\Gamma 1/4}\;\pi}{\cancel{1/4}\;\cancel{\Gamma 1/4}}$$

$$\boxed{\int_0^{\pi/2} \frac{d\theta}{\sqrt{\sin\theta}} \times \int_0^{\pi/2} \sqrt{\sin\theta}\, d\theta = \pi \quad \text{RHS}}$$

3. Evaluate $\int_0^{\pi/2} \sqrt{\cot\theta}\, d\theta$ by expressing it in terms of Gamma function.

$$\rightarrow \quad I = \int_0^{\pi/2} \sqrt{\cot\theta}\, d\theta = \int_0^{\pi/2} \sqrt{\frac{\cos\theta}{\sin\theta}}\, d\theta$$

$$= \int_0^{\pi/2} \sin^{-1/2}\theta \cdot \cos^{1/2}\theta \; d\theta$$

$$p = -\tfrac{1}{2} \qquad q = \tfrac{1}{2}$$

$$= \frac{1}{2} \beta\left(\frac{-1/2+1}{2}, \frac{1/2+1}{2}\right)$$

$$= \frac{1}{2} \beta\left(1/4, 3/4\right)$$

$$I = \frac{1}{2}\left[\frac{\Gamma 1/4 \ \Gamma 3/4}{\Gamma(1/4+3/4)}\right] = \frac{1}{2}\left[\frac{\Gamma(1/4)\,\Gamma(3/4)}{\Gamma(1)}\right]$$

$$I = \frac{1}{2} \Gamma 1/4 \ \Gamma 3/4$$

$$= \frac{1}{2} \pi\sqrt{2}$$

$$= \frac{\pi\sqrt{2}}{\sqrt{2}\,\sqrt{2}}$$

$$\boxed{\int_0^{\pi/2} \sqrt{\cot\theta}\, d\theta = \frac{\pi}{\sqrt{2}}}$$

Note :- $\int_0^{\pi/2} \sqrt{\tan\theta}\, d\theta = \frac{\pi}{\sqrt{2}}$ (June 2018)