Module 2

Vector Calculus

Scalar and vector field

If to every point (x, y, z) of a region R in space there corresponds a scalar $\phi(x, y, z)$ then ϕ is called a scalar point function and we say that a scalar field ϕ is define in 'R'

Eg: 1) $\phi = x^2 + y^2 + z^2$

2) $\phi = xy^2z^3$

If to every point (x, y, z) of a region R in space there corresponds a vector $\vec{A}(x, y, z)$ then A is called a vector point function and we say that a vector field $\vec{A}$ is define in 'R'

Eg: 1) $\vec{A} = x^2\hat{i} + y^2\hat{j} + z^2\hat{k}$

2) $\vec{A} = xyz\,\hat{i} + yz\hat{j} + z\hat{k}$

Gradient of a scalar field

If ϕ is a scalar function then the gradient of ϕ is vector defined by $\operatorname{grad}\phi = \nabla\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$

Unit normal vector

If $\phi(x, y, z) = C$ represent a surface then the unit normal vector to the surface is defined by $\hat{n} = \frac{\nabla\phi}{|\nabla\phi|}$

Directional derivative

The directional derivative of ϕ along vector $\vec{a}$ is $\nabla\phi \cdot \vec{a}$ the maximum directional derivative is magnitude of $\nabla\phi \cdot |\nabla\phi|$

Divergence of a vector

If $\bar{A} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ then divergence of $\bar{A}$ is scalar defined by $\text{div}\,\bar{A}$ or $\nabla \cdot \bar{A} = \frac{\partial a_1}{\partial x} + \frac{\partial a_2}{\partial y} + \frac{\partial a_3}{\partial z}$

Solenoidal vector

vector $\bar{A}$ is called to be solenoidal if $\text{div}\,\bar{A} = 0$

curl of a vector

If $\bar{A} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ then the curl of a vector is a vector defined by $\text{curl}\,\bar{A}$ or $\nabla \times \bar{A} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_1 & a_2 & a_3 \end{vmatrix}$

$$\text{curl}\,\bar{A} = i\left(\frac{\partial a_3}{\partial y} - \frac{\partial a_2}{\partial z}\right) - j\left(\frac{\partial a_3}{\partial x} - \frac{\partial a_1}{\partial z}\right) + k\left(\frac{\partial a_2}{\partial x} - \frac{\partial a_1}{\partial y}\right)$$

Irrotational vector

A is called to be irrotational vector if $\text{curl}\,\bar{A} = 0$

Problems

If $\phi = x^2y^3z^4$ find $\nabla\phi$, $|\nabla\phi|$ at $(1,-1,1)$

Given $\phi = x^2y^3z^4$

$$\nabla\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$$

$$= \frac{\partial}{\partial x}(x^2y^3z^4)\hat{i} + \frac{\partial}{\partial y}(x^2y^3z^4)\hat{j} + \frac{\partial}{\partial z}(x^2y^3z^4)\hat{k}$$

$$= 2xy^3z^4\hat{i} + 3x^2y^2z^4\hat{j} + 4x^2y^3z^3\hat{k}$$

$$\nabla\phi_{(1,-1,1)} = -2\hat{i} + 3\hat{j} - 4\hat{k}$$

$$|\nabla\phi| = \sqrt{(-2)^2 + (3)^2 + (-4)^2}$$

$$= \sqrt{29}$$

Find the unit normal vector to the surface $xy + yz + zx = c$ at the point $(-1, 2, 3)$

Given

$$\phi = xy + yz + zx - c$$

unit normal vector $\hat{n} = \dfrac{\nabla\phi}{|\nabla\phi|}$

grad ϕ or $\nabla\phi = \dfrac{\partial\phi}{\partial x}\hat{i} + \dfrac{\partial\phi}{\partial y}\hat{j} + \dfrac{\partial\phi}{\partial z}\hat{k}$

$$\nabla\phi = (y+z)\hat{i} + (x+z)\hat{j} + (y+x)\hat{k}$$

$$\nabla\phi_{(-1,2,3)} = 5\hat{i} + 2\hat{j} + \hat{k}$$

$$|\nabla\phi| = \sqrt{5^2 + 2^2 + 1^2}$$

$$|\nabla\phi| = \sqrt{30}$$

unit normal vector

$$\hat{n} = \frac{5\hat{i} + 2\hat{j} + \hat{k}}{\sqrt{30}}$$

$$= \frac{5}{\sqrt{30}}\hat{i} + \frac{2}{\sqrt{30}}\hat{j} + \frac{1}{\sqrt{30}}\hat{k}$$

Find the angle between the directions of the normal to the surface $x^2yz = 1$ at the points $(-1,1,1)$ and $(1,-1,-1)$

Given

$$\phi = x^2yz - 1$$

$$\nabla\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$$

$$\nabla\phi = 2xyz\,\hat{i} + x^2 z\,\hat{j} + x^2 y\,\hat{k}$$

i) $\nabla\phi_{(-1,1,1)} = -2\hat{i} + 1\hat{j} + 1\hat{k}$

$$|\nabla\phi| = \sqrt{(-2)^2 + 1^2 + 1^2}$$

$$|\nabla\phi| = \sqrt{6}$$

$$\hat{n}_1 = \frac{\nabla\phi}{|\nabla\phi|}$$

$$\hat{n}_1 = \frac{-2\hat{i} + 1\hat{j} + 1\hat{k}}{\sqrt{6}}$$

ii) $\nabla\phi_{(1,-1,-1)} = +2\hat{i} - 1\hat{j} - 1\hat{k}$

$$|\nabla\phi| = \sqrt{(+2)^2 + (-1)^2 + (-1)^2}$$

$$|\nabla\phi| = \sqrt{6}$$

$$\hat{n}_2 = \frac{2\hat{i} - 1\hat{j} - 1\hat{k}}{\sqrt{6}}$$

$$\cos\theta = \hat{n}_1 \cdot \hat{n}_2$$

$$= \frac{-2\hat{i} + \hat{j} + \hat{k}}{\sqrt{6}} \cdot \frac{(2\hat{i} - \hat{j} - \hat{k})}{\sqrt{6}}$$

$$= \frac{-4 - 1 - 1}{\cancel{\sqrt{6}}\ 6}$$

$$= -\frac{6}{6}$$

$\cos\theta = -1$ $\qquad\qquad \theta = 180^\circ = \pi$

Find the angle between the surfaces $x \log z = y^2 - 1$ and $x^2 y = 2 - z$ at $(1,1,1)$

Given

$$\phi_1 = x \log z - y^2 + 1$$

$$\nabla \phi_1 = \frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \frac{\partial \phi}{\partial z}\hat{k}$$

$$= \log z\,\hat{i} - 2y\,\hat{j} + \frac{x}{z}\hat{k}$$

$$\nabla \phi_{1\,(1,1,1)} = -2\hat{j} + \hat{k}$$

$$\hat{n}_1 = \frac{\Delta \phi_1}{|\Delta \phi_1|}$$

$$= \frac{-2\hat{j} + \hat{k}}{\sqrt{5}}$$

$$\phi_2 = x^2 y + z - 2$$

$$\nabla \phi_2 = 2xy\,\hat{i} + x^2\hat{j} + \hat{k}$$

$$\nabla \phi_{2\,(1,1,1)} = 2\hat{i} + 1\hat{j} + 1\hat{k}$$

$$\hat{n}_2 = \frac{\Delta \phi_2}{|\nabla \phi_2|}$$

$$= \frac{2\hat{i} + \hat{j} + \hat{k}}{\sqrt{6}}$$

$$\cos\theta = \hat{n}_1 \cdot \hat{n}_2$$

$$= \frac{-2\hat{j}+\hat{k}}{\sqrt{5}} \times \frac{2\hat{i}+\hat{j}+\hat{k}}{\sqrt{6}}$$

$$= \frac{-2+1}{\sqrt{30}}$$

$$\cos\theta = \frac{-1}{\sqrt{30}}$$

$$\theta = \cos^{-1}\left(\frac{-1}{\sqrt{30}}\right)$$

Find the directional ~~and~~ derivative of $\phi = x^2yz + 4xz^2$ at $(1,-2,-1)$ along the vector $\vec{a} = 2\hat{i}-\hat{j}-2\hat{k}$

Given

$$\phi = x^2yz + 4xz^2$$

$$\nabla\phi = (2xyz + 4z^2)\hat{i} + x^2z\hat{j} + (x^2y + 8xz)\hat{k}$$

$$\nabla\phi = 8\hat{i} - \cancel{2}\hat{j} + (-2 - 8)\hat{k}$$

$$\nabla\phi = 8\hat{i} - \cancel{2}\hat{j} - 10\hat{k}$$

$$|\vec{a}| = \sqrt{2^2+1^2+2^2}$$

$$|\vec{a}| = 3$$

Directional ~~vector~~ derivative along $\vec{a}$ is $\nabla\phi \cdot \hat{a}$

$$\nabla\phi \cdot \hat{a} = \nabla\phi \cdot \frac{\vec{a}}{|\vec{a}|}$$

$$= 8\hat{i} - \hat{j} - 10\hat{k} \cdot \frac{(2\hat{i} - \hat{j} - 2\hat{k})}{3}$$

$$\nabla\phi \cdot \hat{a} = \frac{16 + 1 + 20}{3}$$

$$\nabla\phi \cdot \hat{a} = \frac{37}{3}$$

In which direction the directional derivative of x^2yz^3 is maximum at $(2,1,-1)$ and find the magnitude of this direction

$$\phi = x^2yz^3$$

$$\nabla\phi = 2xyz^3\,\hat{i} + x^2z^3\,\hat{j} + 3x^2yz^2\,\hat{k}$$

$$\nabla\phi_{(2,1,-1)} = -4\hat{i} - 4\hat{j} + 12\hat{k}$$

The maximum directional derivative is magnitude of $|\nabla\phi|$

$$|\nabla\phi| = \sqrt{4^2 + 4^2 + 12^2}$$

$$|\nabla\phi| = \sqrt{176}$$

$$|\nabla\phi| = \sqrt{16 \times 11}$$

$|\nabla\phi| = 4\sqrt{11}$ is the required maximum directional derivative

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Find the maximum directional derivative of $\phi = xy + yz + zx$ at $(1,1,1)$

$\cancel{[\phi = xy + yz + zx}$

$\cancel{\nabla\phi = y\hat{i} + z\hat{j} + x\hat{k}}$

$\nabla\phi_{(1,1,1)} \cancel{= \hat{i} + \hat{j} + \hat{k}]}$

$$\nabla\phi = (y+z)\hat{i} + (x+z)\hat{j} + (x+y)\hat{k}$$

$$\nabla\phi_{(1,1,1)} = 2\hat{i} + 2\hat{j} + 2\hat{k}$$

$$|\nabla\phi| = \sqrt{12} = 2\sqrt{3}$$

$|\nabla\phi| = 2\sqrt{3}$ is the required maximum directional derivative

Find the divergence of $\vec{f} = x^3z\,\hat{i} + y^3x\,\hat{j} + z^3y\,\hat{k}$

Given

$$\vec{f} = x^3z\,\hat{i} + y^3x\,\hat{j} + z^3y\,\hat{k}$$

$$\nabla\cdot\vec{f} = \frac{\partial a_1}{\partial x} + \frac{\partial a_2}{\partial y} + \frac{\partial a_3}{\partial z}$$

$$= \frac{\partial}{\partial x}(x^3z) + \frac{\partial}{\partial y}(y^3x) + \frac{\partial}{\partial z}(z^3y)$$

$$\nabla\cdot\vec{f} = 3x^2z + 3y^2x + 3z^2y$$

Show that vector $\vec{f} = 2x^2z\,\hat{i} - 10xyz\,\hat{j} + 3xz^2\,\hat{k}$ is solenoidal

Given

$$\vec{f} = 2x^2z\,\hat{i} - 10xyz\,\hat{j} + 3x^2z^2\,\hat{k}$$

$$\nabla\cdot\vec{f} = 4xz - 10xz + 6x^2z$$

$$\nabla\vec{f} = 0$$

As divergence $\vec{f}$ is 0 we can conclude it is solenoidal vector

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Find the curl $\vec{f}$ given that $\vec{f} = xyz^2\,\hat{i} + xy^2z\,\hat{j} + x^2yz\,\hat{k}$

Given

$$\vec{f} = xyz^2\,\hat{i} + xy^2z\,\hat{j} + x^2yz\,\hat{k}$$

$$\text{curl}\,\vec{f} = i\left(\frac{\partial a_3}{\partial y} - \frac{\partial a_2}{\partial z}\right) - j\left(\frac{\partial a_3}{\partial x} - \frac{\partial a_1}{\partial z}\right) + k\left(\frac{\partial a_2}{\partial x} - \frac{\partial a_1}{\partial y}\right)$$

$$= \hat{i}\left(\frac{\partial x^2yz}{\partial y} - \frac{\partial xy^2z}{\partial z}\right) - \hat{j}\left(\frac{\partial x^2yz}{\partial x} - \frac{\partial xyz^2}{\partial z}\right) +$$

$$\hat{k}\left(\frac{\partial xy^2z}{\partial x} - \frac{\partial xyz^2}{\partial y}\right)$$

$$= \hat{i}\left(x^2z - xy^2\right) - \hat{j}\left(2xyz - 2xyz\right) + \hat{k}\left(y^2z - xz^2\right)$$

$$= \hat{i}\left(x^2z - xy^2\right) + \hat{k}\left(y^2z - xz^2\right)$$

Show that $\bar{f} = (4xy - z^3)\hat{i} + 2x^2\hat{j} - 3xz^2\hat{k}$ is irrotational

Given

$$\bar{f} = (4xy - z^3)\hat{i} + 2x^2\hat{j} - 3xz^2\hat{k}$$

$$\text{curl } \bar{f} = i\left(\frac{\partial a_3}{\partial y} - \frac{\partial a_2}{\partial z}\right) - j\left(\frac{\partial a_3}{\partial x} - \frac{\partial a_1}{\partial z}\right) + k\left(\frac{\partial a_2}{\partial x} - \frac{\partial a_1}{\partial y}\right)$$

$$= i(0-0) - j\left(-3z^2 + 3z^2\right) + k\left(4x - 4x\right)$$

$$= i(0) - j(0) + k(0)$$

$$\text{curl } \bar{f} = 0$$

As $\text{curl } \bar{f} = 0$ we can conclude that the given vector is irrotational.

If $\bar{A} = 2x^2\hat{i} - 3yz\hat{j} + xz^2\hat{k}$ and $\phi = 2z - x^3y$ compute $\bar{A}\cdot\nabla\phi$ and $\bar{A}\times\nabla\phi$ at $(1\ -1\ 1)$

Given

$$\phi = 2z - x^3y$$

$$\nabla\phi = -3x^2y\,\hat{i} - x^3\hat{j} + 2\hat{k}$$

$$\nabla\phi = 3\hat{i} - 1\hat{j} + 2\hat{k}$$

ie ~~$\bar{A}\cdot\nabla\phi$~~ . $\bar{A} = 2x^2\hat{i} - 3yz\hat{j} + xz^2\hat{k}$

$$\bar{A}_{(1\ -1\ 1)} = 2\hat{i} + 3\hat{j} + \hat{k}$$

i) $\vec{A} \cdot \nabla\phi$

$= (2\hat{i} + 3\hat{j} + \hat{k})(3\hat{i} - \hat{j} + 2\hat{k})$

$= 6 - 3 + 2$

$\vec{A} \cdot \nabla\phi = 5$

ii) $\vec{A} \times \nabla\phi = \begin{vmatrix} i & j & k \\ 2 & 3 & 1 \\ 3 & -1 & 2 \end{vmatrix}$

$= i(6+1) - j(4-3) + k(-2-9)$

$= 7\hat{i} - \hat{j} - 11\hat{k}$

Find divergent $\vec{F}$ and curl $\vec{F}$ where $\vec{F} = \nabla(x^3 + y^3 + z^3 - 3xyz)$

Given

$\vec{F} = \nabla(x^3 + y^3 + z^3 - 3xyz)$

Let $\phi = x^3 + y^3 + z^3 - 3xyz$

$\vec{F} = \nabla\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$

$\vec{F} = (3x^2 - 3yz)\hat{i} + (3y^2 - 3xz)\hat{j} + (3z^2 - 3xy)\hat{k}$

$\Delta\vec{F} = \frac{\partial a_1}{\partial x} + \frac{\partial a_2}{\partial y} + \frac{\partial a_3}{\partial z}$

$\Delta\vec{F} = 6x + 6y + 6z$

$$\text{curl}\,\bar{F} = i\left(\frac{\partial a_3}{\partial y} - \frac{\partial a_2}{\partial z}\right) - j\left(\frac{\partial a_3}{\partial x} - \frac{\partial a_1}{\partial z}\right) + k\left(\frac{\partial a_2}{\partial x} - \frac{\partial a_1}{\partial y}\right)$$

$$= \hat{i}\,(-3x + 3x) - \hat{j}\,(-3y + 3y) + \hat{k}\,(-3z + 3z)$$

$$\cancel{6x\hat{i} + 6y\hat{j} - 6z\hat{k}}$$

$$= 0$$

Find divergent $\bar{F}$ and curl $\bar{F}$ where $\bar{F} = \nabla(xy^3z^2)$ at $(1\ -1\ 1)$

$$\bar{F} = \nabla(xy^3z^2)$$

Let $\phi = xy^3z^2$

$$\bar{F} = \nabla\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$$

$$\bar{F} = y^3z^2\,\hat{i} + 3xy^2z^2\,\hat{j} + 2xy^3z\,\hat{k}$$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^3z^2 & 3xy^2z & 2xy^3z \end{vmatrix}$$

$$\Delta\bar{F} = \frac{\partial a_1}{\partial x} + \frac{\partial a_2}{\partial y} + \frac{\partial a_3}{\partial z}$$

$$\Delta\bar{F} = 6xyz^2 + 2xy^3$$

$$\Delta\bar{F} = -6 - 2$$

$$\Delta\bar{F} = -8$$

$$\text{curl}\,\bar{F} = i\left(\frac{\partial a_3}{\partial y} - \frac{\partial a_2}{\partial z}\right) - j\left(\frac{\partial a_3}{\partial x} - \frac{\partial a_1}{\partial z}\right) + k\left(\frac{\partial a_2}{\partial x} - \frac{\partial a_1}{\partial y}\right)$$

$$= i\,(6xy^2z - 6xy^2z) - j\,(2y^3z - 2y^3z) +$$

$$k\,(3y^2z^2 - 3y^2z^2)$$

$$= \underline{0}$$

If $\vec{A} = xz^3\hat{i} - 2x^2yz\hat{j} + 2yz^4\hat{k}$ find $\nabla\cdot\vec{A}$, $\nabla\times\vec{A}$ and $\nabla\cdot(\nabla\times\vec{A})$ at $(1\ -1\ 1)$.

Given

$$\vec{A} = xz^3\hat{i} - 2x^2yz\hat{j} + 2yz^4\hat{k}$$

$$\cancel{\nabla\cdot\vec{A} = (z^3 - 4xyz)\hat{i} + (-2x^2z + 2z^4)\hat{j} + (3xz^2 - 2x^2y + 8yz^3)}$$

$$\cancel{\nabla\cdot\vec{A} = (1+4)\hat{i} + (\ )}$$

$$\nabla\vec{A} = z^3 - 2x^2z + 8yz^3$$

$$\nabla\vec{A} = 1 - 2 - 8$$

$$\nabla\vec{A} = -9$$

ii) $\nabla\times\vec{A}$

$$\text{curl}\ \vec{A} = i\left(\frac{\partial a_3}{\partial y} - \frac{\partial a_2}{\partial z}\right) - j\left(\frac{\partial a_3}{\partial x} - \frac{\partial a_1}{\partial z}\right) + k\left(\frac{\partial a_2}{\partial x} - \frac{\partial a_1}{\partial y}\right)$$

$$= i(8z^3 + 2x^2) - j(-3z^2) + k(-4xz - 0)$$

$$= i(8+2) - j(-3) + k(-4)$$

$$= 10\hat{i} + 3\hat{j} - 4\hat{k}$$

$= i\left(2z^4 + 2x^2y\right) - j\left(-3xz^2\right) + k\left(-4xyz - 0\right)$

$= i(2-2) - j(-3) + k(4)$

$= 3\hat{j} + 4\hat{k}$

$\nabla \cdot (\nabla \times \vec{A}) = \dfrac{\partial a_1}{\partial x} + \dfrac{\partial a_2}{\partial y} + \dfrac{\partial a_3}{\partial z}$

$\nabla \cdot (\nabla \times \vec{A}) = \dfrac{\partial}{\partial y}(3) + \dfrac{\partial}{\partial z}(4)$

$\nabla \cdot (\nabla \times \vec{A}) = 0$

Find the unit normal to the surface $x^2y - 2xz + 2y^2z^4 = 10$ at $(2, 1, -1)$

Let

$\phi = x^2y - 2xz + 2y^2z^4 - 10$

$\nabla\phi = (2xy - 2z)\hat{i} + (x^2 + 4yz^4)\hat{j} + (-2x + 8y^2z^3)\hat{k}$

$\nabla\phi = (4+2)\hat{i} + (4+4)\hat{j} + (-4 \cancel{4} - 8)\hat{k}$

$\nabla\phi = 6\hat{i} + 8\hat{j} - 12\hat{k}$ $\qquad \dfrac{\cancel{2}(3\hat{i} + 4\hat{j} - 6\hat{k})}{\cancel{2}\sqrt{61}}$ $\qquad$ 144 + 64 + 36 = 244

$|\nabla\phi| = \sqrt{244}$

$|\nabla\phi| = 2\sqrt{61}$

unit normal vector

$\hat{n} = \dfrac{\nabla\phi}{|\nabla\phi|} = \dfrac{\cancel{2}(3\hat{i} + 4\hat{j} - 6\hat{k})}{\cancel{2}\sqrt{61}}$

Find the unit normal to the surface $xy^3z^4 = 4$ at $(-1,-1,2)$

Let

$$\phi = xy^3z^4 - 4$$

$$\nabla\phi = y^3z^4\,\hat{i} + 3xy^2z^4\,\hat{j} + 4xy^3z^3\,\hat{k}$$

$$\nabla\phi_{(-1,-1,2)} = (-1)^3(2)^4\,\hat{i} + 3(-1)(-1)^2(2)^4\,\hat{j} + 4(-1)(-1)^3(2)^3$$

$$\nabla\phi_{(-1,-1,2)} = -16\,\hat{i} - 48\,\hat{j} + 32\,\hat{k}$$

32×32
64
96×
1024

256
9
1024

$$|\nabla\phi| = \cancel{16}\,(\hat{i} + 3\hat{j} - 2\hat{k})$$

$$|\nabla\phi| = \sqrt{14}$$

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|}$$

$$= \frac{\hat{i} + 3\hat{j} - 2\hat{k}}{\sqrt{14}}$$

Find the angle between the surfaces $x^2+y^2+z^2=9$ and $z=x^2+y^2-3$ at $(2,-1,2)$

Let $\phi_1 = x^2+y^2+z^2-9$

$$\nabla\phi_1 = 2x\hat{i}+2y\hat{j}+2z\hat{k}$$

$$\nabla\phi_1(2,-1,2) = 4\hat{i}-2\hat{j}+4\hat{k}$$

$$|\nabla\phi_1| = \sqrt{16+4+16}$$

$$= \sqrt{36}$$

$$= 6$$

$$\hat{n}_1 = \frac{\nabla\phi_1}{|\nabla\phi_1|} = \frac{4\hat{i}-2\hat{j}+4\hat{k}}{6}$$

$$\hat{n}_1 = \frac{2(2\hat{i}-\hat{j}+2\hat{k})}{6}$$

$$\hat{n}_1 = \frac{2\hat{i}-\hat{j}+2\hat{k}}{3}$$

Let $\phi_2 = x^2+y^2-3-z$

$$\nabla\phi_2 = 2x\hat{i}+2y\hat{j}-\hat{k}$$

$$\nabla\phi_2(2,-1,2) = 4\hat{i}-2\hat{j}-\hat{k}$$

$$|\nabla\phi_2| = \sqrt{16+4+1}$$

$$|\nabla\phi_2| = \sqrt{21}$$

$$\hat{n}_2 = \frac{4\hat{i}-2\hat{j}-\hat{k}}{\sqrt{21}}$$

Angle between the surfaces

$\cos\theta = \hat{n}_1 \cdot \hat{n}_2$

$= \dfrac{2\hat{i} - \hat{j} + 2\hat{k}}{3} \cdot \dfrac{4\hat{i} - 2\hat{j} - \hat{k}}{\sqrt{21}}$

$\cos\theta = \dfrac{8 + 2 - 2}{3\sqrt{21}}$

$\cos\theta = \dfrac{8}{3\sqrt{21}}$

$\theta = \cos^{-1}\left(\dfrac{8}{3\sqrt{21}}\right)$

Find the angle between the surfaces $x^2+y^2-z^2=4$ and $z = x^2+y^2-13$ at $(2,1,2)$

Let $\phi_1 = x^2+y^2-z^2-4$

$\nabla\phi_1 = 2x\hat{i} + 2y\hat{j} - 2z\hat{k}$

$\nabla\phi_{1\,(2,1,2)} = 4\hat{i} + 2\hat{j} - 4\hat{k}$

$|\nabla\phi_1| = \sqrt{16+4+16}$

$|\nabla\phi_1| = 6$

$\hat{n}_1 = \dfrac{\Delta\phi_1}{|\nabla\phi_1|}$

$\hat{n}_1 = \dfrac{2(2\hat{i} + \hat{j} - 2\hat{k})}{6}$

$\hat{n}_1 = \dfrac{(2\hat{i} + \hat{j} - 2\hat{k})}{3}$

Let $\phi_2 = x^2 + y^2 - z - 13$

$$\nabla \phi_2 = 2x\hat{i} + 2y\hat{j} - 1\hat{k}$$

$$\nabla \phi_{2\,(2\,1\,2)} = 4\hat{i} + 2\hat{j} - \hat{k}$$

$$|\nabla \phi_2| = \sqrt{16+4+1}$$

$$|\nabla \phi_2| = \sqrt{21}$$

$$\hat{n}_2 = \frac{4\hat{i} + 2\hat{j} - \hat{k}}{\sqrt{21}}$$

$$\cos\theta = \hat{n}_1 \cdot \hat{n}_2$$

$$\cos\theta = \frac{(2\hat{i} + \hat{j} - 2\hat{k})}{3} \frac{(4\hat{i} + 2\hat{j} - \hat{k})}{\sqrt{21}}$$

$$\cos\theta = \frac{8+2+2}{3\sqrt{21}} \qquad \theta = \cos^{-1}\frac{4}{\sqrt{21}}$$

$$\cos\theta = \frac{4}{\sqrt{21}}$$

Find the directional derivative of $\phi = 4xz^3 - 3x^2y^2z$ at $(2\ {-1}\ 2)$ along $2\hat{i} - 3\hat{j} + 6\hat{k}$

Given

$$\phi = 4xz^3 - 3x^2y^2z$$

$$\nabla \phi = 4z^3 - 6xy^2z\,\hat{i} + (-6x^2yz)\hat{j} + (12xz^2 - 3x^2y^2)\hat{k}$$

$$\nabla \phi_{(2\ -1\ 2)} = 4(2)^3 - 6(2)(-1)^2(2)\hat{i} + (-6(2)(-1)(2))\hat{j} + 12(2)(2)^2 - 3(2)^2(-1)^2\,\hat{k}$$

$$= 8\hat{i} + 48\hat{j} + 84\hat{k}$$

also we have

$$\vec{a} = 2\hat{i} - 3\hat{j} + 6\hat{k}$$

$$|\bar{a}| = \sqrt{4+9+36}$$
$$= \sqrt{49}$$
$$= 7$$

$$D.D = \nabla\phi \cdot \frac{\bar{a}}{|\bar{a}|}$$

$$= (8\hat{i} + 48\hat{j} + 84\hat{k}) \cdot \frac{(2\hat{i} - 3\hat{j} + 6k)}{7}$$

$$= \frac{16 - 144 + 504}{7}$$

$$= \frac{376}{7}$$

Find the D.D of $\phi = x^2yz^3$ at $(1,1,1)$ along $\hat{i} + \hat{j} + 2\hat{k}$

Given
$$\phi = x^2yz^3$$
$$\nabla\phi = 2xyz^3\hat{i} + x^2z^3\hat{j} + 3x^2yz^2\hat{k}$$
$$\nabla\phi = 2\hat{i} + 1\hat{j} + 3\hat{k}$$

also we have
$$\bar{a} = \hat{i} + \hat{j} + 2\hat{k}$$
$$|\bar{a}| = \sqrt{6}$$

$$DD = \nabla\phi \cdot \frac{\bar{a}^r}{|\bar{a}^r|}$$

$$= 2\hat{i} + 1\hat{j} + 3\hat{k} \cdot \frac{(\hat{i} + \hat{j} + 2\hat{k})}{\sqrt{6}}$$

$$= \frac{2 + 1 + 6}{\sqrt{6}}$$

$$= \frac{9}{\sqrt{6}}$$

Find the directional derivative of $\phi = \frac{xz}{x^2+y^2}$ at $(1\ -1\ 1)$ along $\hat{i} - 2\hat{j} + \hat{k}$

Given

$$\phi = \frac{xz}{x^2+y^2}$$

$$\nabla\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$$

$$= \frac{(x^2+y^2)z - xz\cdot 2x}{(x^2+y^2)^2}\hat{i} - \frac{(x^2+y^2)(0) - xz \times 2y}{(x^2+y^2)^2}\hat{j} + \frac{x}{x^2+y^2}\hat{k}$$

$$= \frac{x^2z + y^2z - 2x^2z}{x^2+y^2}\hat{i} - \frac{xz \times 2y}{(x^2+y^2)^2}\hat{j} + \frac{x}{x^2+y^2}\hat{k}$$

$$= \frac{y^2z - x^2z}{x^2+y^2}\hat{i} - \frac{2xyz}{(x^2+y^2)}\hat{j} + \frac{x}{x^2+y^2}\hat{k}$$

$$\nabla\phi_{(1\ -1\ 1)} = \frac{1-1}{(1+1)^2}\hat{i} + \frac{2}{1+1}\hat{j} + \frac{1}{1+1}\hat{k}$$

$$\nabla\phi = \frac{2}{4}\hat{j} + \frac{1}{2}\hat{k}$$

$$\nabla\phi = \frac{1}{2}\hat{j} + \frac{1}{2}\hat{k}$$

also we know

$$\bar{a} = \hat{i} - 2\hat{j} + k$$

$$|\bar{a}| = \sqrt{1^2+2^2+1^2}$$

$$|\bar{a}| = \sqrt{6}$$

$$D.D = \nabla\phi \cdot \frac{\bar{a}}{|\bar{a}|}$$

$$= \left(\frac{1}{2}\hat{j} + \frac{1}{2}\hat{k}\right) \cdot \frac{(\hat{i} - 2\hat{j} + \hat{k})}{\sqrt{6}}$$

$$= \frac{-1 + \frac{1}{2}}{\sqrt{6}}$$

$$= -\frac{1}{2\sqrt{6}}$$

$\phi = xy^3 + yz^3$ at $(2\ -1\ 1)$ along $\hat{i} + 2\hat{j} + 2\hat{k}$

Show that $\vec{F} = (y+z)\hat{i} + (z+x)\hat{j} + (x+y)\hat{k}$ is irrotational also find scalar function ϕ such that $\vec{F} = \nabla\phi$

Given

$$\vec{F} = (y+z)\hat{i} + (z+x)\hat{j} + (x+y)\hat{k}$$

If curl $\vec{F} = 0$ irrotation

$$\text{curl}\,\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y+z & z+x & x+y \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}(x+y) - \frac{\partial}{\partial z}(z+x)\right] - \hat{j}\left[\frac{\partial}{\partial x}(x+y) - \frac{\partial}{\partial z}(y+z)\right] + \hat{k}\left[\frac{\partial}{\partial x}(z+x) - \frac{\partial}{\partial y}(y+z)\right]$$

$$= \hat{i}(1-1) - \hat{j}(1-1) + \hat{k}(1-1)$$

$$\text{curl}\,\vec{F} = 0$$

$\therefore \vec{F}$ is irrotational

To find ϕ such that $\vec{F} = \nabla\phi$

$$\vec{F} = (y+z)\hat{i} + (z+x)\hat{j} + (x+y)\hat{k}$$

$$\nabla\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$$

Equating the corresponding components and integrating

$$\frac{\partial\phi}{\partial x} = y+z \qquad \therefore \phi = \int (y+z)\,dx + \phi_1(y,z)$$

$$\phi = xy + xz + \phi_1(y,z)$$

$$\frac{\partial \phi}{\partial y} = z + x \qquad \therefore \phi = \int (z+x)\, dy + \phi_2(z,x)$$

$$\phi = yz + xy + \phi_2(z,x)$$

$$\frac{\partial \phi}{\partial z} = x + y \qquad \therefore \phi = \int (x+y)\, dz + \phi_3(x,y)$$

$$\phi = xz + yz + \phi_3(x,y)$$

Let us choose $\phi_1(y,z) = yz$, $\phi_2(z,x) = xz$,

$\phi_3(x,y) = xy$

$$\therefore \phi = xy + xz + yz$$

$$\nabla \phi = \vec{f}$$

Show that $\vec{f} = 2xyz^3\hat{i} + x^2z^3\hat{j} + 3x^2yz^2\hat{k}$ is irrotational and find ϕ such that $\vec{f} = \nabla\phi$

Given

$$\vec{f} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xyz^3 & x^2z^3 & 3x^2yz^2 \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}(3x^2yz^2) - \frac{\partial}{\partial z}(x^2z^3)\right] - \hat{j}\left[\frac{\partial}{\partial x}(3x^2yz^2) - \frac{\partial}{\partial z}(2xyz^3)\right]$$

$$+\hat{k}\left[\frac{\partial}{\partial x}(x^2z^3) - \frac{\partial}{\partial y}(2xyz^3)\right]$$

$$= \hat{i}(3x^2z^2 - 3x^2z^2) - \hat{j}(6xyz^2 - 6xyz^2) + \hat{k}(2xz^3 - 2xz^3)$$

curl $\vec{f}^{Y} = 0$

$\therefore \vec{f}^{Y}$ is irrotational

To find of ϕ such that $\vec{f}^{Y} = \nabla\phi$

$$\vec{f}^{Y} = 2xyz^3\,\hat{i} + x^2z^3\,\hat{j} + 3x^2yz^2\,\hat{k}$$

$$\nabla\phi = \frac{\partial}{\partial x}\phi\,\hat{i} + \frac{\partial\phi}{\partial y}\,\hat{j} + \frac{\partial\phi}{\partial z}\,\hat{k}$$

Evaluating the corresponding components and integrating

$$\frac{\partial\phi}{\partial x} = 2xyz^3 \quad \therefore \phi = \int (2xyz^3)\,dx + \phi_1(x,y,z)$$

$$\phi = x^2yz^3 + \phi_1(x,y,z)$$

$$\frac{\partial\phi}{\partial y} = x^2z^3 \qquad \therefore \phi = \int (x^2z^3)\,dy + \phi_2(x,z)$$

$$\phi = \frac{yx^2}{2}z^3 + \phi_2(x,z)$$

$$\frac{\partial\phi}{\partial z} = 3x^2yz^2 \quad \therefore \phi = \int (3x^2yz^2)\,dz + \phi_3(x,y,z)$$

$$\phi = x^2yz^3 + \phi_3(x,y,z)$$

$$\phi = x^2yz^3$$

Show that $\vec{f} = (z + \sin y)\hat{i} + (x\cos y - z)\hat{j} + (x-y)\hat{k}$ is irrotational hence find the scalar function ϕ such that $\vec{f} = \nabla\phi$

Given

$$\vec{f} = (z + \sin y)\hat{i} + (x\cos y - z)\hat{j} + (x-y)\hat{k}$$

$$\vec{f} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (z+\sin y) & (x\cos y - z) & (x-y) \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}(x-y) - \frac{\partial}{\partial z}(x\cos y - z)\right] - \hat{j}\left[\frac{\partial}{\partial x}(x-y) - \frac{\partial}{\partial z}(z+\sin y)\right]$$

$$+\hat{k}\left[\frac{\partial}{\partial x}(x\cos y - z) - \frac{\partial}{\partial y}(z+\sin y)\right]$$

$$= \hat{i}\left[-1+1\right] - \hat{j}\left[1-1\right] + \hat{k}\left[\cos y - \cos y\right]$$

$\text{curl}\,\vec{f} = 0$

$\therefore$ the given vector $\vec{f}$ is irrotational

To find ϕ such that $\vec{f} = \nabla\phi$

$$\vec{f} = (z+\sin y)\hat{i} + (x\cos y - z)\hat{j} + (x-y)\hat{k}$$

$$\nabla\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$$

Evaluating the corresponding components and integrating

$\frac{\partial \phi}{\partial x} = (z + \sin y) \quad \therefore \phi = \int (z + \sin y)\, dx + \phi_1 (z, y)$

$\phi = zx + x \sin y + \phi_1 (z, y)$

$\frac{\partial \phi}{\partial y} = (x \cos y - z) \quad \therefore \phi = \int (x \cos y - z) dy + \phi_2 (x, y, z)$

$\phi = x \sin y - zy + \phi_2 (x, z)$

$\frac{\partial \phi}{\partial z} = (x - y) \quad \therefore \phi = \int (x - y)\, dz + \phi_3 (x, y)$

$\phi = zx - yz + \phi_3 (x, y)$

Let us choose

$\phi_1 (y, z) = -zy \quad \phi_2 (x, z) = xz \quad \phi_3 (x, y) = x \sin y$

$\therefore \underline{\underline{\phi = x \sin y - zy + xz}}$

~~Show that $\vec{f} = axy$~~.

Find the value of constant a such that the vector field $\vec{f} = (axy - z^3)\hat{i} + (a-2) x^2 \hat{j} + (1-a) xz^2 \hat{k}$ is irrotational and hence ~~the~~ find a scalar function ϕ such that $\vec{f} = \nabla \phi$

Given

$\vec{f}$ is irrotational

$\therefore \text{curl } \vec{f} = 0$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (axy - z^3) & (a-2)x^2 & (1-a)xz^2 \end{vmatrix} = 0$$

$$\hat{i}\left[\frac{\partial}{\partial y}(1-a)xz^2 - \frac{\partial}{\partial z}(a-2)x^2\right] - \hat{j}\left[\frac{\partial}{\partial x}(1-a)xz^2 - \frac{\partial}{\partial z}(axy - z^3)\right]$$

$$+\hat{k}\left[\frac{\partial}{\partial x}(a-2)x^2 - \frac{\partial}{\partial y}(axy - z^3)\right] = 0$$

$$\hat{i}\left[0 - 0\right] - \hat{j}\left[(z^2 - az^2) + 3z^2\right] + \hat{k}\left[(2ax - 4x) - ax\right] = 0$$

$$-\hat{j}\left[+4z^2 - az^2\right] + \hat{k}\left[ax - 4x\right] = 0$$

$$-\hat{j}\, z^2\left[4 - a\right] + \hat{k}\, x\left[a - 4\right] = 0$$

$$\therefore \cancel{a = 0} \quad a = 4$$

To find ϕ such that $\vec{F} = \nabla\phi$

$$\vec{F} = (4xy - z^3)\,\hat{i} + 2x^2\hat{j} - 3xz^2\hat{k}$$

$$\nabla\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$$

Evaluating the corresponding components and integrating

$\frac{\partial}{\partial x} = (4xy - z^3) \quad \therefore \phi = \int (4xy - z^3)\, dx + \phi_1(y,z)$

$\phi = 2x^2y - xz^3 + \phi_1(y,z)$

$\frac{\partial}{\partial y} = 2x^2 \quad \therefore \phi = \int 2x^2\, dy + \phi_2(x,z)$

$\phi = 2x^2y + \phi_2(x,z)$

$\frac{\partial}{\partial z} = -3xz^2 \quad \therefore \phi = \int (-3xz^2)\, dz + \phi_3(x,y)$

$\phi = -\frac{3xz^3}{3}$

$\phi = -xz^3 + \phi_3(x,y)$

Let us choose

$\phi_1(y,z) = 0 \quad \phi_2(x,z) = -xz^3 \quad \phi_3(x,y) = 2x^2y$

$\phi = 2x^2y - xz^3$

Find the constant a and b such that $\bar{F} = (axy + z^3)\hat{i} + (3x^2 - z)\hat{j} + (bxz^2 - y)\hat{k}$ is irrotational also find a scalar function ϕ such that $\bar{F} = \nabla\phi$

Given

$\bar{F}$ is irrotational

$\therefore \text{curl}\,\bar{F} = 0$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (axy+z^3) & (3x^2-z) & (bxz^2-y) \end{vmatrix} = 0$$

$$\hat{i}\left[\frac{\partial}{\partial y}(bxz^2-y) - \frac{\partial}{\partial z}(3x^2-z)\right] - \hat{j}\left[\frac{\partial}{\partial x}(bxz^2-y) - \frac{\partial}{\partial z}(axy+z^3)\right]$$

$$+\hat{k}\left[\frac{\partial}{\partial x}(3x^2-z) - \frac{\partial}{\partial y}(axy+z^3)\right] = 0$$

$$\hat{i}\left[-1+1\right] - \hat{j}\left[bz^2 - 3z^2\right] + \hat{k}\left[6x - ax\right] = 0$$

$$-\hat{j}\left[bz^2 - 3z^2\right] + \hat{k}\left[6x - ax\right] = 0$$

$$-\hat{j}\, z^2\left[b-3\right] + \hat{k}\, x\left[6-a\right] = 0$$

$$\therefore \quad b = 3 \qquad a = 6$$

To find ϕ such that $\vec{f} = \nabla\phi$

$$\vec{f} = (6xy + z^3)\hat{i} + (3x^2 - z)\hat{j} + (3xz^2 - y)\hat{k}$$

$$\nabla\phi = \frac{\partial}{\partial x}\phi\,\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial}{\partial z}\phi\,\hat{k}$$

Evaluating the corresponding components and integrating

$\frac{\partial}{\partial x}(6xy+z^3)$ $\quad \therefore \phi = \int (6xy+z^3)\,dx + \phi_1(y,z)$

$\phi = 6x^2y + xz^3 + \phi_1(y,z)$

$\frac{\partial}{\partial y}(3x^2-z)$ $\quad \therefore \phi = \int (3x^2-z)\,dy + \phi_2(x,z)$

$\phi = 3x^2y - zy + \phi_2(x,z)$

$\frac{\partial}{\partial z}(3xz^2-y)$ $\quad \therefore \phi = \int (3xz^2-y)\,dz + \phi_3(xy)$

$\phi = xz^3 - zy + \phi_3(x,y)$

$\phi_1(y,z) = -zy$

$\phi_2(x,z) = 3x^2y + xz^3$

$\phi_3(x,y) = 3x^2y$

$\phi = 6x^2y + xz^3 - zy$

Curvilinear co-ordinates

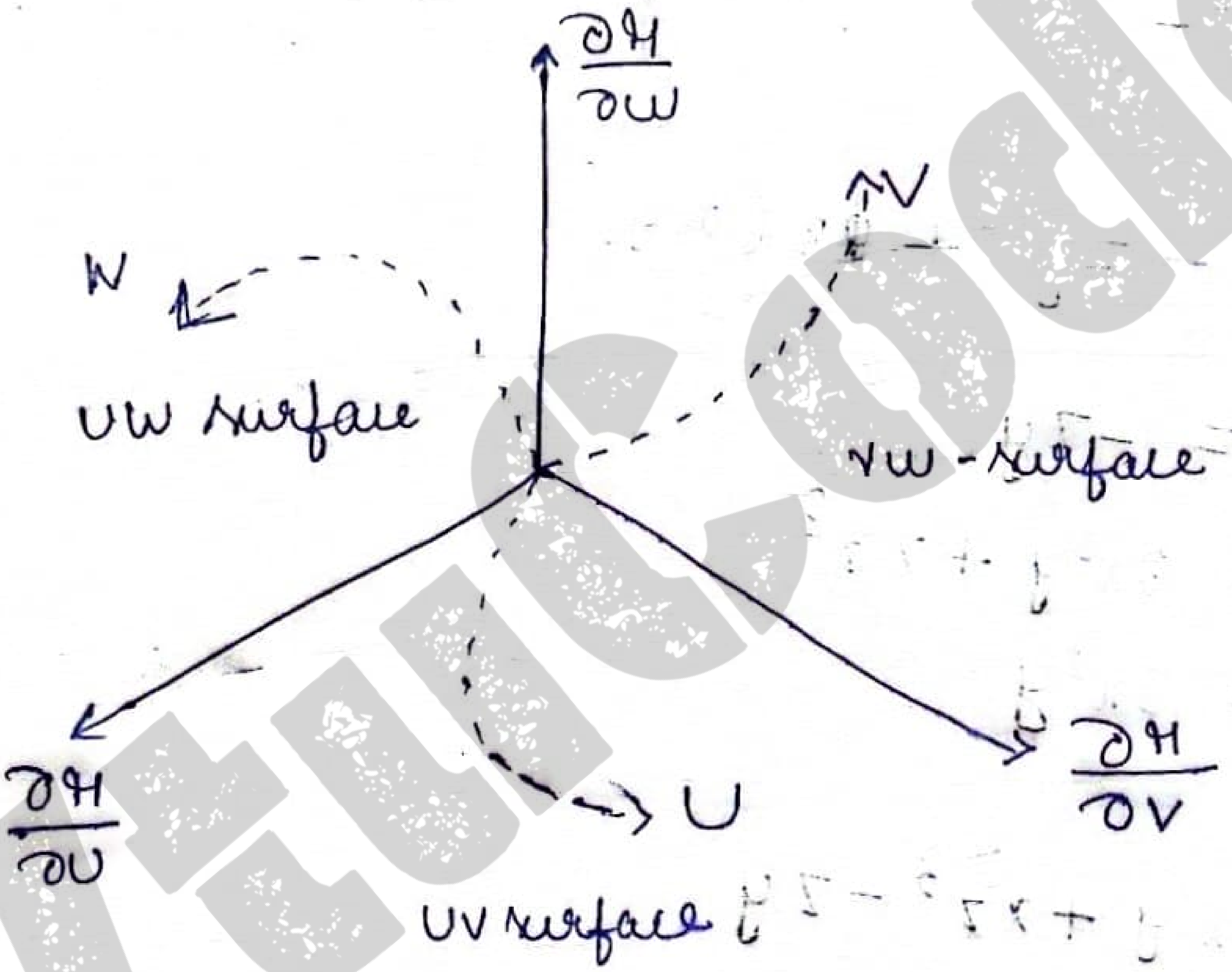
Let the co-ordinates of any point P in space be (x,y,z) in the cartesian system.

Suppose x,y,z are expressible in terms of new co-ordinates (u_1, u_2, u_3) then x,y,z are functions of $u_1\ u_2\ u_3$.

To express $u_1\ u_2\ u_3$ in terms of $x\ y\ z$ by solving or eliminating then the co-ordinates (u_1, u_2, u_3) are known as curvilinear co-ordinates of the point P where it is assumed that the correspondence between (x, y, z) & (u_1, u_2, u_3) is unique

orthogonal curvilinear co-ordinates

A system of curvilinear co-ordinates is said to be orthogonal if at each point the tangents to the co-ordinate curves are mutually perpendicular



Scale factor, base vectors and orthogonality

Let $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ be position vector of a point in space, we have $\vec{r} = \vec{r}(u_1, u_2, u_3)$

$\frac{\partial \vec{r}}{\partial u_1}, \frac{\partial \vec{r}}{\partial u_2}, \frac{\partial \vec{r}}{\partial u_3}$ → Tangent vectors

$h_1 = \left| \frac{\partial \vec{r}}{\partial u_1} \right| \quad h_2 = \left| \frac{\partial \vec{r}}{\partial u_2} \right| \quad h_3 = \left| \frac{\partial \vec{r}}{\partial u_3} \right|$ → Scale factor

$$\hat{e}_1 = \frac{\frac{\partial \bar{r}}{\partial u_1}}{\left|\frac{\partial \bar{r}}{\partial u_1}\right|} = \frac{1}{h_1}\frac{\partial \bar{r}}{\partial u_1}$$

$$\hat{e}_2 = \frac{\frac{\partial \bar{r}}{\partial u_2}}{\left|\frac{\partial \bar{r}}{\partial u_2}\right|} = \frac{1}{h_2}\cdot\frac{\partial \bar{r}}{\partial u_2}$$

$$\hat{e}_3 = \frac{\frac{\partial \bar{r}}{\partial u_3}}{\left|\frac{\partial \bar{r}}{\partial u_3}\right|} = \frac{1}{h_3}\frac{\partial \bar{r}}{\partial u_3}$$

base vectors

orthogonality of the curvilinear co-ordinates system

$\hat{e}_1\cdot\hat{e}_2 = 0 \qquad \hat{e}_2\cdot\hat{e}_3 = 0 \qquad \hat{e}_3\cdot\hat{e}_1 = 0$

$\hat{e}_1\times\hat{e}_2 = \hat{e}_3 \qquad \hat{e}_2\times\hat{e}_3 = \hat{e}_1 \qquad \hat{e}_3\times\hat{e}_1 = e_2$

Transformation between cartesian and curvilinear systems.

Let $x = u \quad y = v \quad z = w$

$\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$\bar{r} = u\hat{i} + \cancel{y}v\hat{j} + w\hat{k}$

$$\frac{\partial \bar{r}}{\partial x} = \frac{\partial \bar{r}}{\partial u} = \hat{i}, \quad \frac{\partial \bar{r}}{\partial y} = \frac{\partial \bar{r}}{\partial v} = \hat{j}, \quad \frac{\partial \bar{r}}{\partial z} = \frac{\partial \bar{r}}{\partial w} = \hat{k}$$

the scale factors are

$$h_1 = \left|\frac{\partial \bar{r}}{\partial u}\right| = |\hat{i}| = 1 \quad h_2 = \left|\frac{\partial \bar{r}}{\partial v}\right| = |\hat{j}| = 1 \quad h_3 = \left|\frac{\partial \bar{r}}{\partial w}\right| = |\hat{k}| = 1$$

The base vectors are

$$\hat{e}_1 = \frac{1}{h_1}\frac{\partial \overline{r}}{\partial U} = \hat{i}$$

$$\hat{e}_2 = \frac{1}{h_2}\frac{\partial \overline{r}}{\partial V} = \hat{j}$$

$$\hat{e}_3 = \frac{1}{h_3}\frac{\partial \overline{r}}{\partial V} = \hat{k}$$

For cartesian co-ordinate system

$h_1 = h_2 = h_3 = 1 \qquad \hat{e}_1 = \hat{i}, \hat{e}_2 = \hat{j}, \hat{e}_3 = \hat{k}$

also we have

$e_1 \cdot e_2 = 0 \qquad e_2 \cdot e_3 = 0 \qquad e_3 \cdot e_1 = 0$

$e_1 \times e_2 = e_3 \qquad e_2 \times e_3 = e_1 \qquad e_3 \times e_1 = e_2$

Thus the cartesian co-ordinates are orthogonal

Scale factors of the cylindrical and spherical system

cylindrical system

cylindrical polar co-ordinates (ρ, ϕ, z) given by the transformation

$x = \rho\cos\phi, \quad y = \rho\sin\phi, \quad z = z$

Let $\overline{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$\overline{r} = \rho\cos\phi\,\hat{i} + \rho\sin\phi\,\hat{j} + z\hat{k}$

By definition of scale factors

$$h_1 = \left|\frac{\partial \overline{r}}{\partial \rho}\right| = \left|\cos\phi\,\hat{i} + \sin\phi\,\hat{j} + 0\right|$$

$$= \sqrt{\cos^2\phi + \sin^2\phi} = \sqrt{1} = 1$$

$h_1 = 1$

$$h_2 = \left|\frac{\partial \bar{r}}{\partial \phi}\right| = \left| -\rho \sin\phi \hat{i} + \rho \cos\phi \hat{j} + 0 \right|$$

$$\sqrt{\rho^2 \sin^2\phi + \rho^2 \cos^2\phi}$$

$$= \sqrt{\rho^2 (\sin^2\phi + \cos^2\phi)}$$

$$= \rho$$

$$h_2 = \rho$$

$$h_3 = \left|\frac{\partial \bar{r}}{\partial z}\right| = \left| 1\hat{k} \right|$$

$$= \sqrt{1}$$

$$= 1$$

$\therefore h_1 = 1 \quad h_2 = \rho \quad h_3 = 1$ are the scale factors of cylindrical system

Spherical system

spherical polar co-ordinates (r, θ, ϕ) given by the transformation

$x = r\sin\theta\cos\phi \qquad y = r\sin\theta\sin\phi \qquad z = r\cos\theta$

Let $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$$\bar{r} = r\sin\theta\cos\phi\,\hat{i} + r\sin\theta\sin\phi\,\hat{j} + r\cos\theta\,\hat{k}$$

By definition of scale factor.

$$h_1 = \left|\frac{\partial \bar{r}}{\partial r}\right| = \left| \sin\theta\cos\phi\hat{i} + \sin\theta\sin\phi\hat{j} + \cos\theta\hat{k} \right|$$

$$\sqrt{\sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi + \cos^2\theta}$$

$$= \sqrt{\sin^2\theta(\cos^2\phi + \sin^2\phi) + \cos^2\theta}$$

$$= \sqrt{\sin^2\theta + \cos^2\theta}$$

$$= 1$$

$$h_2 = \frac{\partial \bar{r}}{\partial \theta} = \left| r\cos\theta\cos\phi\,\hat{i} + r\cos\theta\sin\phi\,\hat{j} - r\sin\theta\,\hat{k} \right|$$

$$= \sqrt{r^2\cos^2\theta\cos^2\phi + r^2\cos^2\theta\sin^2\phi + r^2\sin^2\theta}$$

$$= \sqrt{r^2\cos^2\theta(\cos^2\phi + \sin^2\phi) + r^2\sin^2\theta}$$

$$= \sqrt{r^2\cos^2\theta + r^2\sin^2\theta}$$

$$= \sqrt{r^2(\cos^2\theta + \sin^2\theta)}$$

$$= r$$

$$h_3 = \frac{\partial \bar{r}}{\partial \phi} = \left| -r\sin\theta\sin\phi\,\hat{i} + r\sin\theta\cos\phi\,\hat{j} + 0 \right|$$

$$= \sqrt{r^2\sin^2\theta\sin^2\phi + r^2\sin^2\theta\cos^2\phi}$$

$$= \sqrt{r^2\sin^2\theta(\sin^2\phi + \cos^2\phi)}$$

$$= \sqrt{r^2\sin^2\theta}$$

$h_3 = r\sin\theta$

$\therefore h_1 = 1 \quad h_2 = r \quad h_3 = r\sin\theta$ are the factors of spherical system

orthogonality of cylindrical system

For cylindrical system

$\vec{r} = \rho\cos\phi\,\hat{i} + \rho\sin\phi\,\hat{j} + z\,\hat{k}$

Let e_ρ, e_ϕ, e_z be the base vectors

$$e_\rho = \frac{1}{h_1}\frac{\partial \vec{r}}{\partial \rho} = \cos\phi\,\hat{i} + \sin\phi\,\hat{j} + 0$$

$$e_\phi = \frac{1}{h_2}\frac{\partial \vec{r}}{\partial \phi} = \frac{1}{\rho}\left[-\rho\sin\phi\,\hat{i} + \rho\cos\phi\,\hat{j} + 0\right]$$

$$e_z = \frac{1}{h_3}\frac{\partial \vec{r}}{\partial z} = \hat{k}$$

consider $e_\rho \cdot e_\phi = -\sin\phi\cos\phi + \sin\phi\cos\phi$

$$e_\rho \cdot e_\phi = 0$$

consider $e_\phi \cdot e_z = 0$

consider $e_z \cdot e_\rho = 0$

Thus the cylindrical system is orthogonal

Orthogonality of spherical system

For spherical system

$\vec{r} = r\sin\theta\cos\phi\,\hat{i} + r\sin\theta\sin\phi\,\hat{j} + r\cos\theta\,\hat{k}$

Let the base vectors be $\hat{e}_r, \hat{e}_\theta, \hat{e}_\phi$

$$\hat{e}_r = \frac{1}{h_1}\frac{\partial \vec{r}}{\partial r} = \frac{1}{1}\left[\sin\theta\cos\phi\,\hat{i} + \sin\theta\sin\phi\,\hat{j} + \cos\theta\,\hat{k}\right]$$

$$\hat{e}_\theta = \frac{1}{h_2}\frac{\partial \vec{r}}{\partial \theta} = \frac{1}{r}\left[r\cos\theta\cos\phi\,\hat{i} + r\cos\theta\sin\phi\,\hat{j} - r\sin\theta\,\hat{k}\right]$$

$$\hat{e}_\theta = \cos\theta\cos\phi\,\hat{i} + \cos\theta\sin\phi\,\hat{j} - \sin\theta\,\hat{k}$$

$$\hat{e}_\phi = \frac{1}{h_3}\frac{\partial \vec{r}}{\partial \phi} = \frac{1}{r\sin\theta}\left[-r\sin\theta\sin\phi\,\hat{i} + r\sin\theta\cos\phi\,\hat{j} + 0\right]$$

$$\hat{e}_\phi = -\sin\phi\,\hat{i} + \cos\phi\,\hat{j}$$

Consider

$$\hat{e}_r \cdot \hat{e}_\theta = \sin\theta\cos\theta\cos^2\phi + \sin\theta\cos\theta\sin^2\phi - \sin\theta\cos\theta$$

$$\hat{e}_r \cdot \hat{e}_\theta = \sin\theta\cos\theta(\cos^2\phi + \sin^2\phi) - \sin\theta\cos\theta$$

$$\hat{e}_r \cdot \hat{e}_\theta = \sin\theta\cos\theta - \sin\theta\cos\theta$$

$$\hat{e}_r \cdot \hat{e}_\theta = 0$$

Consider

$$\hat{e}_\theta \cdot \hat{e}_\phi = -\sin\phi\cos\phi\cos\theta + \sin\phi\cos\phi\cos\theta$$

$$\hat{e}_\theta \cdot \hat{e}_\phi = 0$$

Consider

$$\hat{e}_\phi \cdot \hat{e}_r = -\sin\phi\sin\theta\cos\phi + \cos\phi\sin\theta\sin\phi$$

$$\hat{e}_\phi \cdot \hat{e}_r = 0$$