Module-4

Numerical method I

Numerical solution of Algebraic and transcendental equation

Algebraic equation

Equation involving algebraic quantities like $x, x^2, x^3, \ldots$ etc are called as algebraic equation

Eg: $x^4 + x^3 = 80$

$x^3 - 4x + 9 = 0$

transcendental equation

equations involving non-algebraic quantities like e^x, $\log x$, $\sin x$, $\tan x$ etc... are called as transcendental equation.

Eg: $xe^x - 2 = 0$

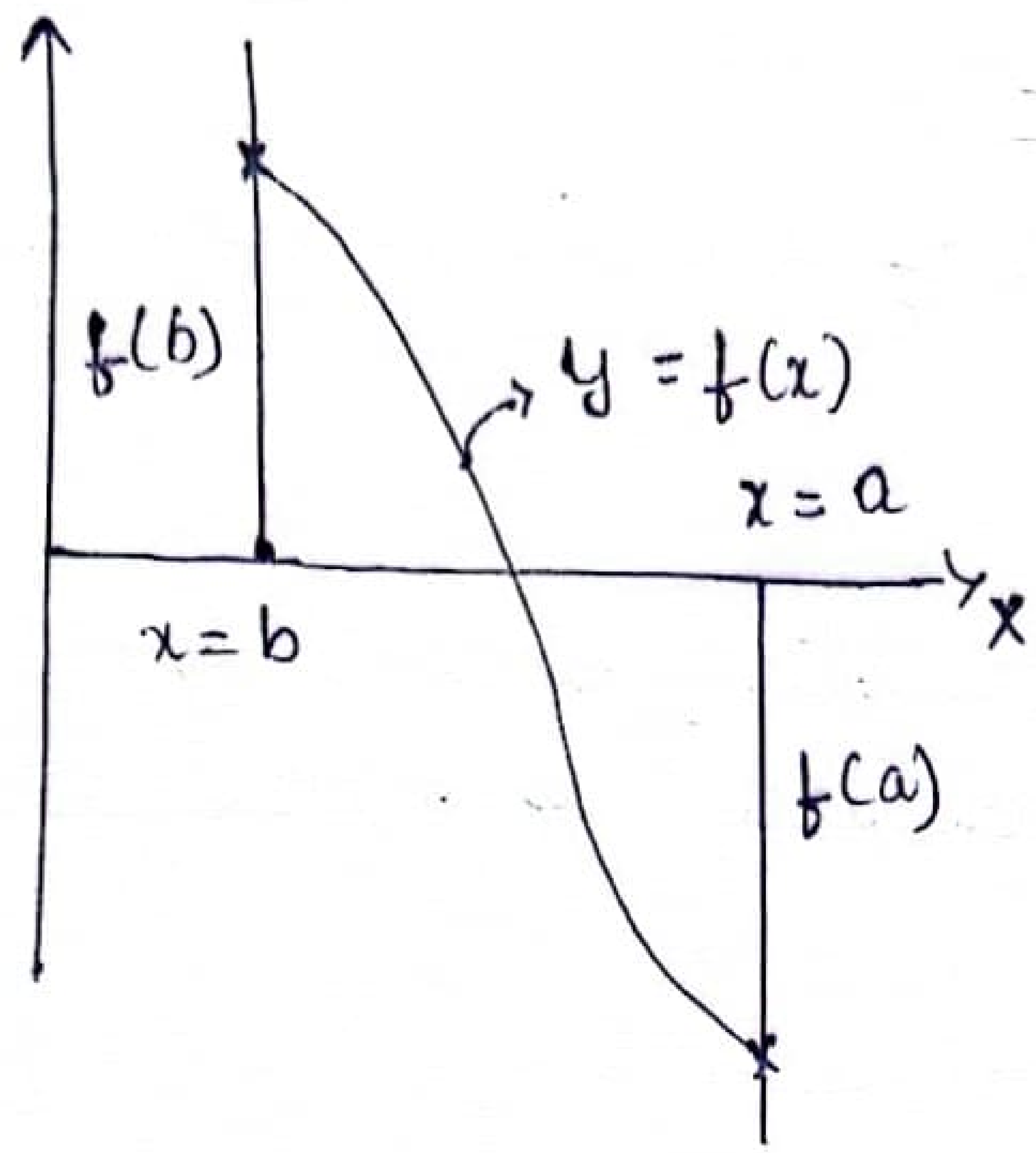
$x \log_e x - 12 = 0$

Iteration process

Numerical methods of finding approximate roots of the given equation is a repetative process known as iteration process.

Fundamental properties

If $f(x) = 0$ is a real valued continuous function of the real variable x. we have the following fundamental property. If there exists two values a, b such that $f(x)$ has opposite signs, say $f(a) < 0$, $f(b) > 0$ equivalently $f(a) f(b) < 0$, then there always exist atleast one real root in the interval (a, b)



there are two numerical iterative method

1. Newton - Raphson method
2. Regula - Falsi method

The first approximation to the root x_0 is given by

$$\boxed{x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}}$$

The second approximation is obtained by replacing x_0 by x_1 in RHS of the above expression

In General newton - Raphson iteration formula is given by

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$$

This procedure is repeated upto two consecutive approximated roots became equal

Problems

1. find a real root of the equation $x^3+x^2+3x+4=0$ by applying newton raphson method.

Let $f(x) = x^3+x^2+3x+4$

$f(0) = 4 > 0 \quad f(-1) = 1 > 0 \quad f(-2) = -6 < 0$

$\therefore$ The real root lies between $(-1,-2)$

Let $x_0 = -1$ $\quad f'(x) = 3x^2+2x+3$

N-R formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

I iteration $(n=0)$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$x_1 = -1 - \frac{1}{4}$$

$$x_1 = \frac{-5}{4}$$

$$x_1 = -1.25$$

II iteration $n=1$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$x_2 = -1.25 - \frac{f(-1.25)}{f'(-1.25)}$$

$3(1.5625) + (2(-1.25)) + 3$

$= 5.1875$

$(-1.25)^3 + (-1.25)^2 + 3(-1.25) + 4$

$= -1.95 + 1.5625 - 3.75 + 4$

$= -5.7 + 5.5625$

$= -0.1375$

$$x_2 = -1.25 - \frac{[(-1.25)^3 + (-1.25)^2 + 3(-1.25) + 4]}{[3(-1.25)^2 + 2(-1.25) + 3]}$$

$$x_2 = -1.2229 \approx 1.223$$

III approximation $n = 2$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$= -1.223 - \frac{[(-1.223)^3 + (-1.223)^2 + 3(-1.223) + 4]}{[3(-1.223)^2 + 2(-1.223) + 3]}$$

$$= -1.2225$$

$\therefore$ the required real root is -1.2225

2. Use newton raphson method to find a real root of the equation $x^3 - 2x - 5 = 0$ correct to three decimal places

Let $f(x) = x^3 - 2x - 5 = \cancel{0}$

$f(0) = -5 < 0$ $\quad f(1) = -6$ $\quad f(2) = -1$ $\quad f(3) = 16$

$\therefore$ the real root lies b/w $(-1, 16)$

3) Use newton raphson method to find a real roots of $x\sin x + \cos x = 0$ carry out the iteration unto 4 decimal places of accuracy for the given $x = \pi$

Let $f(x) = x\sin x + \cos x$

$f'(x) = x\cos x + \sin x - \sin x$

$f'(x) = x\cos x$

Let $x_0 = \pi$

N.P formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

I approximation $(n=0)$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= \pi - \frac{f(\pi)}{f'(\pi)}$$

$$= \pi - \frac{(\pi\sin\pi + \cos\pi)}{\pi\cos\pi}$$

$$= \pi - \frac{(-1)}{-3.14}$$

$$= \pi - \frac{1}{3.14}$$

$$= \frac{8.8596}{3.14}$$

$= 2.8233$

II approximation $n = 1$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$x_2 = 2.8233 - \frac{(2.8233 \sin 2.8233 + \cos 2.8233)}{(2.8233 \cos 2.8233)}$$

$$x_2 = 2.8233 - \cancel{0.8835}\ (-0.06623)$$

$$x_2 = \underline{2.7986}$$

III approximation $n = 2$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$x_3 = 2.7986 - \frac{[2.7986 \sin 2.7986 + \cos 2.7986]}{2.7986 \cos 2.7986}$$

$$x_3 = 2.7984$$

IV approximation $n = 3$

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)}$$

$$x_4 = 2.7984 - \frac{[2.7984 \sin 2.7984 + \cos 2.7984]}{2.7984 \cos 2.7984}$$

$$x_4 = 2.7984$$

$\therefore$ the required real root is 2.7984

4) Find the real root of the equation $xe^x - 2 = 0$ correct ~~two~~ - three decimals using N-R method

Let
$f(x) = xe^x - 2$ $\qquad f' = xe^x + e^x$

$f(0) = -2 < 0$ $\quad f(1) = 0.7183 > 0$

$\therefore$ the real root $(0\ 1)$

N-R formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

~~$x_1 = 1 - 0 \frac{(-2)}{1}$~~ $\quad x_1 = 1 - \frac{0.7183}{5.43}$

~~$x_1 = 3)$~~ $\quad x_1 = 0.8679$

II approximation $(n=1)$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$x_2 = 0.8679 - \frac{(0.8679\, e^{0.8679} - 2)}{(0.8679\, e^{0.8679} + e^{0.8679})}$$

$x_2 = 0.8528$

III approximation $(n=2)$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$x_3 = 0.8528 - \frac{(0.8528e^{0.8528} - 2)}{(0.8528e^{0.8528} + e^{0.8528})}$$

$$x_3 = 0.8526$$

$\therefore$ the real root is $\underline{0.8526}$

~~IV approximation~~

Using N-R method find the approximated root of the equation $x \log_{10} x = 1.2$ correct to four decimal places that near 2.5.

Let

$$f(x) = x \log_{10} x - 1.2$$

$$f'(x) = x \times \frac{1}{x} \log_{10} e + \log_{10} x$$

$$\cancel{f'(x) = 1 + \log_{10} x}$$

$$f'(x) \cdot \log_{10} e + \log_{10} x$$

$$f'(x) = 0.4343 + \log_{10} x$$

N-R formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

I iteration

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$x_1 = 2.5 - \frac{f(2.5)}{f'(2.5)}$$

$$x_1 = 2.5 - \frac{(2.5\log_{10}2.5 - 1.2)}{0.4343 + \log_{10}2.5}$$

$$x_1 = 2.7465$$

II iteration

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$x_2 = 2.7465 - \frac{(2.7465\log_{10}2.7465 - 1.2)}{(0.4343 + \log_{10}2.7465)}$$

$$x_2 = 2.7406$$

III iteration

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$= (2.7406) - \frac{(2.7406\log_{10}2.7406 - 1.2)}{(0.4343 + \log_{10}2.7406)}$$

$$x_3 = 2.74064$$

$\therefore$ The required real root is 2.7406.

By newton-raphson method find the real root of $x\tan x + 1 = 0$ which is near to $x = 2.5$

Let $f(x) = x\tan x + 1$

~~$f(0) = 0\tan 0 + 1$~~ $\quad f'(x) = x\sec^2 x + \tan x$.

I iteration

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$x_1 = 2.5 - \frac{f(2.5)}{f'(2.5)}$$

2.0055

$(1 \div \cos 2.5)^2$

$$= 2.5 - \frac{(2.5\tan 2.5 + 1)}{(2.5\sec^2 2.5 + \tan 2.5)}$$

$x_1 = 2.7755$

II iteration

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$x_2 = 2.7755 - \frac{(2.7755\tan 2.7755 + 1)}{(2.7755\sec^2 2.7755 + \tan 2.7755)}$$

$x_2 = 2.7983$

III iteration

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$x_3 = 2.7983 - \frac{(2.7983 \tan 2.7983 + 1)}{(2.7983 \sec^2 2.7983 + \tan 2.7983)}$$

$$x_3 = 2.7983$$

$\therefore$ the required real root is 2.7983

using newton raphson method find the real root of $3x = \cos x + 1$ near $x=0.5$ upto 4 decimal

$x_1 = 0.6085$
$x_2 = 0.6071$
$x_3 = 0.6071$

Let $f(x) = 3x - \cos x - 1$

$f'(x) = 3 + \sin x$

I

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$x_1 = 0.5 - \frac{f(0.5)}{f'(0.5)}$$

$$x_1 = 0.5 - \frac{(3(0.5) - \cos 0.5 - 1)}{3 + \sin 0.5}$$

$$x_1 = 0.6085$$

Regula falsi method or method of falsi position

To find a numerical solution of the equation $f(x) = 0$ by regula falsi method or method of falsi position

Let the root of the equation $f(x) = 0$ lies b/w a and b then $f(a)$ and $f(b)$ are of opposite signs therefore the I[st] approximation of the root is given by

$$\boxed{x = \frac{af(b) - bf(a)}{f(b) - f(a)}}$$

This procedure is repeated upto two consecutive approximate root become equal

Problem

Use the regula falsi method to find the real root of the equation $x^3 - 2x - 5 = 0$ correct to three decimal places.

Let

$f(x) = x^3 - 2x - 5$

$f(0) = -5 < 0$ $\quad f(1) = -6 < 0$ $\quad f(2) = -1 < 0$,

$f(3) = 16 > 0$

since -1 is closer to 0 we take the initial approximation closer to 2

$f(2.1) = 0.061 > 0$

~~$f(a) = 2$~~ $\quad a = 2 \quad b = 2.1$

$$x = \frac{af(b) - bf(a)}{f(b) - f(a)}$$

lies b/w $(2, 2.1)$

$$x = \frac{2 \times 0.061 - 2.1(-1)}{0.061 + 1}$$

$x_1 = 2.0942$

$f(x_1) = (2.0942)^3 - 2(2.0942) - 5$

$f(x_1) = -0.00392 < 0$

II iteration

$$x_2 = \frac{2.0942 \times 0.061 - 2.1(-0.00392)}{0.061 - (-0.00392)}$$

$x_2 = 2.09455$

$\therefore$ the required real root is 2.094

Find the real root of the equation $\tan x + \tanh x = 0$ lies between 2 and 3 by using regula falsi method carry out three iteration

Let

$f(x) = \tan x + \tanh x$

$f(2) = -1.221 < 0$

$f(3) = 0.8525 > 0$

I iteration

$$x_1 = \frac{af(b) - bf(a)}{f(b) - f(a)}$$

$$x_1 = \frac{2(-1.221) - 3(0.8525)}{0.8525 - (-1.221)}$$

$x_1 = 2.59$

$f(x_1) = f(2.59)$

$= \tan(2.59) + \tanh(2.59)$

$= 0.3735 > 0$

$\therefore$ the roots lies b/w

$(2, 2.59)$

<u>II</u> iteration

$$x_2 = \frac{af(b) - b(f(a)}{f(b) - f(a)}$$

$$x_2 = \frac{2(0.3735) - 2.59(-1.221)}{(0.3735 - (-1.221))}$$

$x_2 = 2.4518$

$f(x_2) = f(2.4518)$

$= 0.1603 > 0$

$\therefore$ the roots lies b/w $(2, 2.4518)$

<u>III</u> iteration

$$x_3 = \frac{af(b) - bf(a)}{f(b) - f(a)}$$

$$x_3 = \frac{2(0.1603) - 2.4518(-1.221)}{0.1603 - (-1.221)}$$

$x_3 = 2.3994$ //

$\therefore$ the required 3rd iteration is $\underline{\underline{2.3994}}$

3. Find the real root of equation $2x + \log_{10}x = 7$ b/w 3.5 & 4 using regula falsi method

Let
$$f(x) = 2x + \log_{10}x - 7$$

$$f(3.5) = -0.5441$$
$$f(4) = \cancel{1.6020}\ 0.3979$$

I iteration

$$x_1 = \frac{af(b) - bf(a)}{f(b) - f(a)}$$

$$= \frac{\cancel{3.5(1.6020) - 4(0.5440)}}{\cancel{1.6020 - 0.5440}}$$

$$= \frac{3.5(0.3979) - 4(-0.5441)}{0.3979 - (-0.5441)}$$

$$= 3.7889$$

$$f(3.7889) = 0.0013640$$

$\therefore$ the roots lies b/w $(3.5, 3.79)$

II iteration

$$x_2 = \frac{3.5(0.00136) - 3.79(-0.5441)}{(0.00136 - 0.5441)}$$

$$x_2 = 3.79$$

∴ the required root is 3.79

Find the smallest positive root of the equation $x^2 - \log_e x = 12$ by regula falsi method

Let $f(x) = x^2 - \log_e x - 12$

$f(0) = -12 < 0$

$f(1) = -11 < 0$

$f(2) = -8.69 < 0$

$f(3) = -4.09 < 0$

$f(4) = 2.613 > 0$

∴ the root lies b/w (3,4) the root lies in neighbourhood of 4

$f(3.6) = -0.3209$

$f(3.7) = 0.3817$

$a = 3.6$

$b = 3.7$

I iteration

$$x_1 = \frac{a f(b) - b f(a)}{f(b) - f(a)}$$

$$= \frac{3.6(0.3817) - 3.7(-0.3209)}{0.3817 - (-0.3209)}$$

$$= 3.6457$$

$f(3.6457) < 0$

~~f(x)~~ $= -0.00242 < 0$

$\therefore$ the roots lies b/w

$(3.6457, 3.7)$

II iteration

$$x_2 = \frac{af(b) - bf(a)}{f(b) - f(a)}$$

$$= \frac{3.6457(-0.3817) - 3.7(-0.00242)}{(0.3817 - (-0.00242))}$$

$x_2 = 3.6460$

Find the root of the equation $xe^x - \cos x = 0$, using regula falsi method. H.W

Finite difference

Let $y = f(x)$ be the function, in this function x is independent variable and y is dependent variable. by assigning a value for x, we get a corresponding value $y = f(x)$ for the function. the value assigned to x is called the argument. The corresponding value of the function is called entry the difference between two consecutive value of x is called the interval of difference it is denoted by h

Forward difference

Let $y = f(x)$ be a function and h is the interval of difference, then $f(x+h) - f(x)$ is called the first forward difference of the function between the arguments x and $x+h$. It is denoted by $\Delta f(x)$

$$\Delta f(x) = f(x+h) - f(x)$$

$$\Delta f(x_0) = f(x_0+h) - f(x_0) \; ; \; \Delta y_0 = y_1 - y_0$$

$$\Delta f(x_1) = f(x_1+h) - f(x_1) \; ; \; \Delta y_1 = y_2 - y_1$$

$$\vdots$$

$$\Delta f(x_{n-1}) = f(x_{n-1}+h) - f(x_{n-1}) \; ; \; \Delta y_{n-1} = y_n - y_{n-1}$$

The difference of first forward differences is called second forward differences

$$\Delta^2 y_0 = \Delta y_1 - \Delta y_0$$

$$\Delta^2 y_1 = \Delta y_2 - \Delta y_1$$

$$\vdots$$

$$\Delta^2 y_{n-2} = \Delta y_{n-1} - \Delta y_{n-2}$$

Forward difference Table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\cdots$	$\Delta^n y$
x_0	y_0					
		Δy_0				
x_1	y_1		$\Delta^2 y_0$			
		Δy_1		$\Delta^3 y_0$	$\cdots$	$\Delta^n y_0$
x_2	y_2		$\Delta^2 y_1$			
		Δy_2		$\vdots$		
$\vdots$	$\vdots$	$\vdots$	$\vdots$			
				$\Delta^3 y_{n-3}$		
			$\Delta^2 y_{n-2}$			
		Δy_{n-1}				
x_n	y_n					

The first entries in the table Δy_0 $\Delta^2 y_0 \ldots \Delta^n y_0$ are called the leading forward differences

Backward difference

Let $f(x)$ be a given function and h be the difference of interval, then the backward difference ∇ (nabla) is defined as

$$\nabla f(x) = f(x) - f(x-h)$$

$\nabla f(x_n) = f(x_n) - f(x_n - h)$; $\nabla y_n = \nabla y_n - \nabla y_{n-1}$

$\nabla f(x_{n-1}) = f(x_{n-1}) - f(x_{n-1} - h)$; $\nabla y_{n-1} = y_{n-1} - y_{n-2}$

$\vdots$

$\nabla y_2 = y_2 - y_1$

$\nabla y_1 = y_1 - y_0$

Similarly second backward difference are defined by

$$\nabla^2 y_n = \nabla y_n - \nabla y_{n-1}$$

$$\nabla^2 y_{n-1} = \nabla y_{n-1} - \nabla y_{n-2}$$

$$\vdots$$

$$\nabla^2 y_2 = \nabla y_2 - \nabla y_1$$

Backward difference table

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\cdots$	$\nabla^n y$
x_0	y_0					
		∇y_1				
x_1	y_1		$\nabla^2 y_2$	$\nabla^3 y_3$		$\nabla^n y_n$
		∇y_2			$\cdots$	
x_2	y_2		$\vdots$	$\vdots$		
$\vdots$	$\vdots$	$\vdots$		$\nabla^3 y_{n-1}$		
			$\nabla^2 y_{n-1}$			
		∇y_{n-1}		$\nabla^3 y_n$		
			$\nabla^2 y_n$			
x_{n-1}	y_{n-1}	∇y_n				
x_n	y_n					

The last entries in the table $\nabla y_n, \nabla^2 y_n, \nabla^3 y_n, \ldots \nabla^n y_n$ are called the leading backward differences

Interpolation

The method of finding the value y at a specified value of x having known values of x and y is called Interpolation.

Newton - Gregory forward interpolation formula

$$y = y_0 + P\Delta y_0 + \frac{P(P-1)}{2!}\Delta^2 y_0 + \frac{P(P-1)(P-2)}{3!}\Delta^3 y_0 + \text{------}$$

where

$$P = \frac{x - x_0}{h}$$

x → The value of x at which y is to be calculated

x_0 → Initial value of 'x'

h → common difference / step length

Δ → forward difference operator

* Newton - Gregory Backward interpolation formula

$$y = y_n + P\nabla y_n + \frac{P(P+1)}{2!}\nabla^2 y_n + \frac{P(P+1)(P+2)}{3!}\nabla^3 y_n + \ldots$$

where $P = \frac{x - x_n}{h}$

x → The value of x at which y is to be calculated

x_n → final value of 'x'

h → common difference / step length

∇ → Backward difference operator

Note: If the value of x, is near begining of the table then we use newton forward interpolation formula

If the value of x, is near end of table then we use newton backward interpolation formula.

Problems

1) Find $y(0.12)$ given the data

x	0.10	0.15	0.20	0.25	0.30
y	0.1003	0.1511	0.2027	0.2553	0.3033

Since $x = 0.12$ is in the begining of the table we use newton forward interpolation formula

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0.10	0.1003				
		0.0508			
0.15	0.1511		0.0008		
		0.0516		0.0002	
0.20	0.2027		0.0010		0.0002
		0.0526		0.0004	
0.25	0.2553		0.0014		
		~~0.0548~~ 0.0540			
0.30	0.3033				

By NFIF

$$y = y_0 + P\Delta y_0 + \frac{P(P-1)}{2!}\Delta^2 y_0 + \frac{P(P-1)(P-2)}{3!}\Delta^3 y + \frac{P(P-1)(P-2)(P-3)}{4!}\Delta^4 y_0$$

where $P = \dfrac{x - x_0}{h}$

$$\frac{0.12 - 0.10}{0.05}$$

$$= 0.4$$

$$y = 0.1003 + 0.4(0.0508) + \frac{0.4(-0.6)\,0.0008}{2\times1} + \frac{0.4(0.4-1)(0.4-2)}{6}(-0.0002) + \frac{0.4(0.4-1)(0.4-2)(0.4-3)}{24}$$

$$\times 0.0002$$

$$y = 0.1003 + 0.4(0.0508) + 0.2(-0.6)\,0.0008 + \frac{0.4(-0.6)(-1.6)(0.0002)}{6} + \frac{0.4(-0.6)(-1.6)(-2.6)}{24}$$

$$y_{(0.12)} = 0.12053$$

2. Using newton's forward interpolation formula find an interpolating polynomial, that approxima-te function given by the table, hence find $y(0.5)$

x	0	1	2	3	4
y	-5	-10	-9	4	35

x	0	1	2	3	4
y	-5				

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0	-5	-5			
			6		
				6	
1	-10	1			0
			12		
				6	
2	-9	13			
			18		
3	4	31			
4	35				

By NFIF

$$y = y_0 + P\Delta y_0 + \frac{P(P-1)\Delta^2 y_0}{2!} + \frac{P(P-1)(P-2)}{3!}\Delta^3 y_0$$

$\cancel{+ P(P-1)(P-2)(+)}$

$$P = \frac{x - x_0}{h}$$

$$= \frac{x-0}{1}$$

$$= x$$

$$y = -5 + x(-5) + \frac{x(x-1)6}{2} + \frac{x(x-1)(x-2)}{6}6$$

$$y = -5 - 5x + 3x^2 - 3x + x^2 - x(x-2)$$

$$y = -5 - 5x + \cancel{3x^2} - 3x + x^3 - \cancel{2x^2} - \cancel{x^2} + 2x$$

$\cancel{y = -8 - 3x + x^2 x^3}$

$\cancel{y = x^3 - 1}$

$y(x) = x^3 - 6x - 5$

$y(0.5) = (0.5)^3 - 6(0.5) - 5$

$y(0.5) = -7.875$

The Area of a circle (A) corresponding to diameter (D) is given below

D	80	85	90	95	100
A	5026	5674	6362	7088	7854

Find the area corresponding to diameter (105) using an appropriate interpolation formula

since $x = 105$ is at the end of the table we use newton forward interpolation formula

let diameter be x Area is y

x	y	∇y	$\Delta^2 y$	$\nabla^3 y$	$\nabla^4 y$
80	5026				
		648			
85	5674		40	-2	4
		688			
90	6362		38	2	
		726			
			40		
95	7088				
		766			
100	7854				

$$y = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \frac{p(p+1)(p+2)(p+3)}{4!}\nabla^4 y_n$$

$$p = \frac{x - x_n}{h}$$

$$p = \frac{105-100}{5}$$

$$p = 1$$

$$y = 7854 + 1(766) + \frac{1(1+1)}{2}40 + \frac{1(2)(3)}{6}2 + \frac{1(2)(3)(4)}{24}4$$

$$y = 7854 + 766 + 40 + 2 + 4$$

$$y_{(105)} = 8666$$

$\therefore$ The Area corresponding to diameter (105) is 8666.

From the following table the no of students who has obtained

a) less than 45 marks

b) between 40 & 45 marks

c) less than 55 marks

Marks (x)	30-40	40-50	50-60	60-70	70-80
No of students (y)	31	42	51	35	31

Let $y = f(x)$ denotes the no of students who have obtained $<$ x marks

Less than 40 marks = 31

Less than 50 marks = 73

Less than 60 marks = 124

Less than 70 marks = 159

Less than 80 marks = 190

x marks	y no of students	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
40	31				
		42			
50	73		9		
		51		-25	
60	124		-16		37
		35		12	
70	159		-4		
		31			
80	190				

a) By NFIF

$$y = y_0 + P\Delta y_0 + \frac{P(P-1)\Delta^2 y_0}{2!} + \frac{P(P-1)(P-2)\Delta^3 y_0}{3!} +$$

$$\frac{P(P-1)(P-2)(P-3)\Delta^4 y_0}{4!}$$

$$P = \frac{x - x_0}{h} = \frac{45-40}{10} = 0.5$$

$$y = 31 + 0.5(42) + \frac{0.5(0.5-1)9}{2} + \frac{0.5(0.5-1)(0.5-2)}{6}$$

$$\times -25 + \frac{0.5(0.5-1)(0.5-2)(0.5-3)\times 37}{24}$$

$$y = 31 + 21 - 1.125 + 0.0625 \times -25 + (-1.4453)$$

$$y = \underline{\underline{47.8672}} \approx 48$$

b) between 40 and 45

$48 - 31 = 17$

17 students have scored in b/w 40 and 45 marks

c) By NFIF

$$y = y_0 + P\Delta y_0 + \frac{P(P-1)\Delta^2 y_0}{2!} + \frac{P(P-1)(P-2)\Delta^3 y_0}{3!} +$$

$$\frac{P(P-1)(P-2)(P-3)\Delta^4 y_0}{4!}$$

$$p = \frac{x - x_0}{h}$$

$$= \frac{55 - 40}{10}$$

$$= 1.5$$

$$y = 31 + 1.5(42) + \frac{1.5(1.5-1)\,9}{2} + \cancel{1.5(1.5-1)(1.5-1)}$$

$$\frac{1.5(1.5-1)(1.5-2)}{6} \times -25 + \frac{1.5(1.5-1)(1.5-2)(1.5-3)\,37}{24}$$

$$y = 31 + 1.5(42) + 3.375 + 1.5625 + 0.867$$

$$y = 99.8045 \simeq 100$$

A survey conducted in a slum locality reveals the following information as classified below

Income per day (Rs)	under 10	10-20	20-30	30-40	40-50
No of Persons	20	45	115	210	115

Estimate
the probable no of persons in the income group of 20 - 25

Let $y(x)$ denotes the no of persons having income of x Rs

Less than 10 Rs – 20

Less than 20 Rs – 65

Less than 30 Rs – 180

Less than 40 Rs – 390

Less than 50 Rs – 505

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
~~20~~ 10	20				
		45			
~~65~~ 20	65		70		
		115		25	
~~180~~ 30	180		95		−215
		210		−190	
~~390~~ 40	390		−95		
		115			
~~505~~ 50	505				

By NFIF

$$y(25) = y_0 + p\Delta y_0 + \frac{p(p-1)\Delta^2 y_0}{2!} + \frac{p(p-1)(p-2)\Delta^3 y_0}{3!} + \frac{p(p-1)(p-2)(p-3)\Delta^4 y_0}{4!}$$

and $p = \dfrac{x - x_0}{h}$

$$= \frac{25-10}{10}$$

$$= 1.5$$

$$y(25) = 20 + 1.5(20) + \frac{1.5(1.5-1)70}{2} + \frac{1.5(1.5-1)(1.5-2)25}{6}$$

$$+ \frac{1.5(1.5-1)(1.5-2)(1.5-3)}{24}(-215)$$

$$y(25) = 20 + 30 + 1.5 \times 0.5 \times 35 + \frac{1.5 \times 0.5 \times -0.5 \times 25}{6}$$

$$+ \frac{1.5(0.5)(-0.5)(-1.5)(-215)}{24} + 37.5$$

$$y(25) = \cancel{69.648 \approx 70} \; 107.148$$

$$\cancel{70-65} \; 107.148 - 65$$

$$= 42.148$$

The population for a town is given by the following table

Year	1951	1961	1971	1981	1991
Population	19.96	39.65	58.81	77.21	94.61

using newton's forward and backward interpolation formula, calculate the increase in the population from 1955 to 1985

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1951	19.96				
		19.69			
1961	39.65		-0.53		
		19.16		-0.23	
1971	58.81		-0.76		-0.01
		18.4		-0.24	
1981	77.21		-1		
		17.4			
1991	94.61				

By NFIF

$$y(1955) = y_0 + P\Delta y_0 + \frac{P(P-1)\Delta^2 y_0}{2!} + \frac{P(P-1)(P-2)\Delta^3 y_0}{3!}$$

$$+ \frac{P(P-1)(P-2)(P-3)\Delta^4 y_0}{4!}$$

$$p = \frac{x - x_0}{h}$$

$$= \frac{1955 - 1951}{10}$$

$$= \frac{4}{10} = 0.4$$

$$= 19.96 + 0.4(19.69) + \frac{0.4(0.4-1)}{2} - 0.53 +$$

$$\frac{0.4(0.4-1)(0.4-2)}{6} \times -0.23 + \cancel{0.4(0.4-1)(0.4-2)}$$

$$\frac{0.4(0.4-1)(0.4-2)(0.4-3)}{24} \times -0.01$$

$\cancel{4.16 \times 10^{-4}}$

$= 27.885$

By NBIF

$$y(1985) = y_n + P\nabla y_n + \frac{P(P+1)}{2!}\nabla^2 y_n + \frac{P(P+1)(P+2)}{3!}\nabla^3 y_n$$

$$+ \frac{P(P+1)(P+2)(P+3)}{4!}\nabla^4 y_n$$

$$P = \frac{x - x_n}{h}$$

$$= \frac{1985 - 1991}{10}$$

$$= \frac{-6}{10} = -0.6$$

$$y(1985) = 94.61 + (-0.6 \times 17.4) + \frac{-0.6(-0.6+1)}{2} - 1 +$$

$$\frac{-0.6(-0.6+1)(-0.6+2)}{6} - 0.24 +$$

$$\frac{-0.6(-0.6+1)(-0.6+2)(-0.6+3)}{24}(-0.01)$$

$$= 94.61 - 10.44 + \frac{0.6(0.4)}{2} + \frac{0.6(0.4)(1.4)0.21}{6}$$

$$+ \frac{0.6 \times 0.4 \times 1.4 \times 2.4 \times -0.01}{24}$$

$= 84.303$

$\therefore$ the population of the town in 1955 is 27.885 and in 1985 is 84.303

$\therefore$ the population of the town in between 1955 to 1985 is 56.415

$84.303 - 27.885$

$= 56.415$

Given $\sin 45^\circ = 0.7071$, $\sin 50^\circ = 0.7660$ $\sin 55^\circ = 0.8192$ $\sin 60^\circ = 0.8660$ find $\sin 57^\circ$ using an appropriate interpolation formula

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$
45°	0.7071			
		0.0589		
50°	0.7660		-0.0057	
		0.0532		-0.0007
55°	0.8192		-0.0064	
		0.0468		
60°	0.8660			

$$y(57) = y_0 + P\nabla y_n + \frac{P(P+1)\nabla^2 y_n}{2!} + \frac{P(P+1)(P+2)\nabla^3 y_n}{3!}$$

$$P = \frac{x - x_n}{h}$$

$$= \frac{\cancel{40} \; 57-45}{5}$$

$$= \frac{12}{5}$$

$$= 2.4$$

$$= 0.8660 + 2.4(0.0468) + \frac{2.4 \times 3.4 \times -0.0064}{2}$$

$$+ \frac{2.4 \times 3.4 \times 4.4 \times -0.0007}{6}$$

$$y(57) = 0.8387$$

$$\therefore \sin 57^\circ = 0.8387$$

Interpolation when the values of x are not equally spaced

* Divided difference

Let $f(x)$ be the given function $x_0, x_1, x_2 \dots x_n$ are the values of x, which are not equally spaced

Let $f(x_0), f(x_1), \dots f(x_n)$ are the values of function at the point $x_0, x_1, x_2 \dots x_n$ respectively.

The first divided difference between the argument x_0 and x_1 is defined as

$$f(x_0 x_1) = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$f(x_1, x_2) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

$$\vdots$$

$$f(x_{n-1}, x_n) = \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}$$

The second divided difference between the argument x_0 and x_2 is defined as

$$f(x_0 x_2) = \frac{f(x_1, x_2) - f(x_0, x_2)}{x_2 - x_0}$$

Newton divided difference formula

Newton's general interpolation formula is given by

$$y = f(x) = f(x_0) + (x - x_0) f(x_0, x_1) + (x - x_0)(x - x_1) f(x_0 x$$
$$+ (x - x_0)(x - x_1)(x - x_2) f(x_0, x_3) + \dots$$

Problems

1. Given the values

x	5	7	11	13	17
$f(x)$	150	392	1452	2366	5202

using divided difference formula for unequal intervals find $f(9)$

x	$f(x)$	1st D.D	2nd D.D	3rd D.D
5	150	$121 \quad \frac{392-150}{7-5}$	$\frac{265-121}{11-5} = 24$	$\frac{32-24}{13-5} = 1$
7	392	$265 \quad \frac{1452-392}{11-7}$	$\frac{457-265}{13-7} = 32$	$\frac{42-32}{17-7} = 1$
11	1452	$457 \quad \frac{2366-1452}{13-11}$	$\frac{709-457}{17-11}$ 42	
13	2366	$709 \quad \frac{5202-2366}{17-13}$		
17	5202			

By NDDF

$$y = f(x) = f(x_0) + (x-x_0) f(x_0 x_1) + (x-x_0)(x-x_1) f(x_0 x_2)$$
$$+ (x-x_0)(x-x_1)(x-x_2) f(x_0, x_3)$$

$$= 150 + 4(121) + (4)(2)\, 24 +$$
$$(4)(2)(-2)(1)$$

$$f(9) = \underline{\underline{810}}$$

using newtons divided difference interpolation formula find $f(8)$ and $f(15)$ from the given data

x	4	5	7	10	11	13
$f\ y$	48	100	294	900	1210	2028

x $\quad f(x) \quad 1 \quad 2 \quad 2$

x	$f(x)$
4	48
5	100
7	294
10	900
11	1210
13	2028

1

$$\frac{100-48}{5-4}=52$$

$$\frac{294-100}{7-5}=97$$

$$\frac{900-294}{10-7}=202$$

$$\frac{1210-900}{11-10}=310$$

$$\frac{2028-1210}{13-11}=409$$

2

$$\frac{97-52}{7-4}=15$$

$$\frac{202-97}{10-5}=21$$

$$\frac{310-202}{11-7}=27$$

$$\frac{409-310}{13-10}=33$$

3

$$\frac{21-15}{10-4}=1$$

$$\frac{27-21}{11-5}=1$$

$$\frac{33-27}{13-7}=1$$

$y = f(x) = f(x_0) + (x-x_0) f(x_0 x_1) + (x-x_0)(x-x_1)$
$f(x_0 x_2) + (x-x_0)(x-x_1)(x-x_2) f(x_0 x_3)$

$= 48 + 4 \times 52 + 4 \times 3 \times 15 + 4 \times 3 \times 1$
$\times 1$

$= 448$

$f(15) = 48 + 11 \times 52 + 11 \times 10 \times 15 +$
$11 \times 10 \times 8 \times 1$

$= 3150$

using newton divided difference interpolation formula

$f(4)$

x	0	2	3	6
$f(x)$	-4	2	14	158

x	$f(x)$	1	2	3
0	-4	$\frac{2+4}{2-0} = 3$	$\frac{12-3}{3-0} = 3$	$\frac{9-3}{6} =$
2	2	$\frac{14-2}{3-2} = 12$	$\frac{48-12}{6-2} = 9$	
3	14	$\frac{158-14}{6-3} = 48$		
6	158			

$f(4) = -4 + 4(3) + 4 \times 2 \times 3 + 4 \times 2 \times 1 \times 1$

$= 40$

Construct the interpolation polynomial for the data given below using newton's general interpolation formula for divided difference.

x	2	4	5	6	8	10
y	10	96	196	350	868	1746

x	f(x)	1	2
2	10	$\frac{96-10}{4-2} = 43$	$\frac{100-43}{5-2} = 19$
4	96	$\frac{196-96}{5-4} = 100$	$\frac{154-100}{6-4} = 27$
5	196	$\frac{350-196}{6-5} = 154$	$\frac{259-154}{8-5} = 35$
6	350	$\frac{868-350}{8-6} = 259$	$\frac{439-259}{10-6} = 45$
8	868	$\frac{1746-868}{10-8} = 439$	
10	1746		

3

$\frac{27-19}{6-2} = 2$

$\frac{35-27}{8-4} = 2$

$\frac{45-35}{10-5} = 2$

$$y = f(x) = f(x_0) + (x - x_0) f(x_0, x_1) + (x - x_0)(x - x_1)$$

$$f(x_0 x_2) + (x - x_0)(x - x_1)(x - x_2) f(x_0, x_3)$$

$$= 10 + (x-2)\, 43 + (x-2)(x-4)\, 19$$

$$+ (x-2)(x-4)(x-5)\, 2$$

$$= 10 + 43x - 86 + \left[(x^2 - 4x - 2x + 8)\, 19\right]$$

$$+ \left[(x^2 - 4x - 2x + 8)(x-5)\, 2\right]$$

$$= 10 + 43x - 86 + 19x^2 - 76x - 38x + 152$$

$$+ (x^3 - 5x^2 - 4x^2 + 20x - 2x^2 + 10x + 8x - 40)\, 2$$

$$\cancel{= 10 - 86 + 152 - 40 + 43x - 76x - 38x + 20x}$$

$$\cancel{+ 18x + 19x^2 - 5x^2 - 4x^2 - 2x^2 + x^3}$$

$$\cancel{= 36 - 33x + 8x^2 + x^3}$$

$$\cancel{= x^3 + 8x^2 - 33x + 36}$$

$$= 10 + 43x - 86 + 19x^2 - 76x - 38x + 152 +$$

$$2x^3 - 10x^2 - 8x^2 + 40x - 4x^2 + 20x + 16x - 80$$

$$= 2x^3 + 19x^2 - 10x^2 - 8x^2 - 4x^2 + 43x - 76x - 38x$$

$$+ 40x + 20x + 16x + 10 - 86 + 152 - 80$$

$y(x) = 2x^3 - 3x^2 + 5x - 4$

Find the polynomial

H.W

x	0	1	2	5
y	2	3	12	147

Lagrange's interpolation formula

Let $y = f(x)$ be a function

$y_0 = f(x_0), y_1 = f(x_1), y_2 = f(x_2) \ldots y_n = f(x_n)$

are the values of the function for the arguments $x_0, x_1, x_2, \ldots x_n$ which are unequally spaced

$$y = f(x) = \frac{(x-x_1)(x-x_2)\ldots(x-x_n)}{(x_0-x_1)(x_0-x_2)\ldots(x_0-x_n)} y_0 + \frac{(x-x_0)(x-x_2)\ldots(x-x_n)}{(x_1-x_0)(x_1-x_2)\ldots(x_1-x_n)} y_1 + \frac{(x-x_0)(x-x_1)\ldots(x-x_n)}{(x_2-x_0)(x_2-x_1)\ldots(x_2-x_n)} y_2 + \ldots$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_2)\ldots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)(x_n-x_2)\ldots(x_n-x_{n-1})} y_n$$

Note

1. $(x-a)(x-b) = x^2 - (a+b)x + ab$
2. $(x-a)(x-b)(x-c) = x^3 - (a+b+c)x^2 + (ab+bc+ca)x - abc$

Use Lagrange's interpolation formula to fit a polynomial for the data

x	0	1	3	4
y	-12	0	6	12

and hence estimate y at $x = 2$

$$y = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} y_0 +$$

$$\frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} \times y_1 +$$

$$\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} y_2 +$$

$$\frac{(x-x_0)(x-x_1)(x-x_2)\cancel{(x-x_3)}}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} y_3$$

$\cancel{(x^2-x)}(x-4)$

$x^3 \cancel{-4x^2 - x^2 + 4x}$

$$y = f(2) = \frac{(2-1)(-1)(-2)}{(-1)(-3)(-4)} (-12) +$$

$\cancel{(x^2-x)}(x-3)$

$\cancel{x^3-3}$

$$\frac{(2)(-1)(-2)}{1(-2)(-3)} \times 0 +$$

$$\frac{(2)(1)(-2)}{(3)(2)(-1)} \times 6 + \left(\frac{(2)\times(1)\times(-1)}{4\times3\times2} \times 12\right)$$

$= -2 + 4 + -1$

$$= \frac{(x-1)(x-3)(x-4)}{(-1)(-3)(-4)} x - 12 + \frac{x(x-3)(x-4)}{(1)(-2)(-3)} \times 0$$

$$+ \frac{x(x-1)(x-4)}{(3)(2)(-1)} \times 6 + \frac{x(x-1)(x-3)}{4(3)(1)} \times 12$$

$$= x^3 - (1+3+4)x^2 + (3+12+4)x - 12 -$$

$$(x^3 - 5x^2 + 4x) + x^3 - 4x^2 + 3x$$

$$= x^3 - 8x^2 + 19x - 12 - x^3 + 5x^2 - 4x + x^3 - 4x^2 + 3x$$

$$y(x) = x^3 - 7x^2 + 18x - 12$$

$8 - 28 + 36 - 12$

$44 - 40$

at $x = 2$

$$y(2) = 2^3 - 7(2)^2 + 18 \times 2 - 12$$

$$\boxed{y(2) = 4}$$

use Laganges interpolation formula find the value of y at $x = 10$

Given

x	5	6	9	11
y	12	13	14	16

$$f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} y_0 +$$

$$\frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} y_1 +$$

$$\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} y_2 + \frac{x-x_0(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} y_3$$

$x_0 = 5 \quad y_0 = 12$

$x_1 = 6 \quad y_1 = 13$

$x_2 = 9 \quad y_2 = 14$

$x_3 = 11 \quad y_3 = 16$

$$= \cancel{\frac{(4 \times 1 \times -1)}{4 \times 3 \times -2} \times 14 +}$$

$$= \frac{4(1)(-1)}{(-1)(-4)(-6)} \times 12 + \frac{5(1)(-1)}{(1)(-3)(-8)} \times 13 +$$

$$\frac{5 \times 4 \times -1}{(4 \times 3 \times -2)} \times 14 + \frac{5 \times 4 \times 1}{6 \times 5 \times 2} \times 16$$

$$= 2 - \frac{13}{3} + \frac{35}{3} + \frac{16}{3} \qquad \frac{51-13}{3}$$

$$= 2 + \frac{38}{3}$$

$$= \frac{6+38}{3} \qquad \frac{44}{3}$$

$= \cancel{10.66}$ 14.666

$= 14.67$

$f(4)$.

x	0	2	3	6
$f(x)$	-4	2	14	158

$x_0 = 0 \quad y_0 = -4$
$x_1 = 2 \quad y_1 = 2$
$x_2 = 3 \quad y_2 = 14$
$x_3 = 6 \quad y_3 = 158$

$$f(4) = \frac{2 \times 1 \times -2}{-2 \times -3 \times -6} \times -4 + \frac{4 \times 1 \times -2}{2 \times -1 \times -4} \times 2$$

$$+ \frac{4 \times 2 \times -2}{3 \times 1 \times -3} \times 14 + \frac{4 \times 2 \times 1}{6 \times 4 \times 3} \times 158$$

$$= -\frac{4}{18} - 2 + \frac{224}{9} + \frac{158}{9}$$

$$= -0.2222 - 2 + \frac{224}{9} + \frac{158}{9}$$

$$= \boxed{40.222}$$

$$= 40$$

$(-1,0)\ (1\ 2)\ (2,9)\ (3,8)$

hence estimate the value of y when $x = 2.2$

from the given data

$x_0 = -1 \quad x_1 = 1 \quad x_2 = 2 \quad x_3 = 3$

$y_0 = 0 \quad y_1 = 2 \quad y_2 = 9 \quad y_3 = 8$

$$f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} y_0 +$$

$$\frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} y_1 + \frac{(x-x_0)(x-x_1)(x-x_3)\,y_2}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}$$

$$= \frac{(x-1)(x-2)(x-3)}{(-2)(-3)(-4)} \times 0 + \frac{(x+1)(x-2)(x-3)}{(2)(-1)(-2)} \times 2$$

$$+ \frac{(x+1)(x-1)(x-3)}{(3)(1)(-1)} \times 9 + \frac{(x+1)(x-1)(x-2)8}{4(2)(1)}$$

$$= \frac{\left[(x+1)(x^2-3x-2x+6)\right]\times 2}{4} + \frac{(x+1)(x^2-3x-1x+3)}{-3}9$$

$$= \frac{x^3-3x^2-2x^2+6x+x^2-5x+6}{4} - \left[x^3-4x^2+3x+x^2-3x-1x+3\right]\times 3.$$

$$= \frac{(x^3-5x^2+x^2+x+6)}{2} - 3x^2+12x^2-9x-3x^2+9x+3x-9$$

$$y(x) = \frac{1}{2}\left[x^3 - (-1+2+3)x^2 + (-2+6-3)x + 6\right]$$
$$-3\left[x^3 - (-1+1+3)x^2 + (-1+3-3)x + 3\right]$$
$$+ x^3 - (-1+1+2)x^2 + (-1+2-2)x + 2\,]$$

$$= \frac{1}{2}\left[x^3 - 4x^2 + x + 6\right] - 3\left[x^3 - 3x^2 - x + 3\right] +$$
$$x^3 - 2x^2 - x + 2$$

$$= \frac{x^3}{2} - 2x^2 + \frac{x}{2} + 3 - 3x^3 + 9x^2 + 3x - 9 + x^3 - 2x^2 - x + 2$$

$$= 0.5x^3 - 2x^2 + 0.5x + 3 - 3x^3 + 9x^2 + 3x - 9 + x^3 - 2x^2 - x + 2$$

$$= -1.5x^3 + 5x^2 + 2.5x - 4$$

$$= -1.5(2.2)^3 + 5(2.2)^2 + (2.5)(2.2) - 4$$

$$= 9.728$$

Find y at $x = 5$ if $y(1) = -3$, $y(3) = 9$, $y(4) = 30$, $y(6) = 132$

$x_0 = 1 \quad y_0 = -3$

$x_1 = 3 \quad y_1 = 9$

$x_2 = 4 \quad y_2 = 30$

$x_3 = 6 \quad y_3 = 132$

$$y = f(x) = \frac{(2)(1)(-1)}{(-2)(-3)(-5)} \times -3 + \frac{(4)(1)(-1)}{(2)(-1)(-3)} \times 9$$
$$+ \frac{(4)(2)(-1)}{(3)(1)(-2)}\, 30 + \frac{(4)(2)(1)}{(5)(3)(2)} \times 132$$

$$= \frac{-2}{10} + -\frac{36}{16} + \frac{8 \times 30}{6} + \frac{4 \times 132}{15}$$

$$= \underline{69}$$

Given the values of x and y

x	2.8	4.1	4.9	6.2
y	9.8	13.4	15.5	19.6

find y when $x = 8$ using Lagrange's interpolation formula

$x_0 = 2.8$ $\quad y_0 = 9.8$

$x_1 = 4.1$ $\quad y_1 = 13.4$

$x_2 = 4.9$ $\quad y_2 = 15.5$

$x_3 = 6.2$ $\quad y_3 = 19.6$

$$y = f(x) = \frac{(3.9)(3.1)(2.8)}{(-1.3)(-2.1)(-3.4)} \times 9.8 + \frac{(5.2)(3.1)(1.8)}{(1.3)(-0.8)(-2.1)} \times 13.4 + \frac{(5.2)(3.9)(1.8)}{(2.1)(0.8)(-1.3)} \times 15.5 +$$

$$\frac{(5.2)(3.9)(3.1)}{(3.4)(2.1)(1.3)} \times 19.6$$

$y(8) = 28.7336$

Numerical Integration

This is the process of obtaining approximate value of the definite integral $I = \int_a^b y\,dx$ without actually integrating the function, but only using values of y at some points of x equally spaced over $[a, b]$

There are three rules to obtain the value of the definite integral numerically

I Rule : Trapezoidal Rule

$$\int_{x_0}^{x_0+nh} y\,dx = \frac{h}{2}\left[y_0 + 2(y_1+y_2+y_3+\dots+y_{n-1}) + y_n\right]$$

II Rule : Simpson's $\frac{1}{3}$rd Rule

$$\int_{x_0}^{x_0+nh} y\,dx = \frac{h}{3}\left[(y_0+y_n) + 4(y_1+y_3+\dots+y_{n-1}) + 2(y_2+y_4+\dots+y_{n-2})\right]$$

III Rule : Simpson's $3/8$th rule

$$\int_{x_0}^{x_0+nh} y\,dx = \frac{3h}{8}\left[(y_0+y_n) + 3(y_1+y_2+y_4+\dots) + 2(y_3+y_6+y_9+\dots)\right]$$

Note: * $h \rightarrow$ common difference

* here n is the number of sub intervals of the interval

* If there are n intervals then $(n+1)$ no of co-ordinates

Problems

1. Evaluate $\int_2^7 \frac{dx}{x}$ using trapezoidal rule taking $n=5$

Given

$x_0 = 2 \quad x_0 + nh = 7 \quad n = 5$

$nh = 5$

$h = 1$

x	2	3	4	5	6	7
y	0.5	0.3333	0.25	0.2	0.166	0.1428

$\frac{1}{x}\, dx$

$$\int_{x_0}^{x_0+nh} y\,dx = \frac{h}{2}\left[y_0 + 2(y_1 + y_2 + y_3 + \cdots y_{n-1}) + y_n\right]$$

$$\int_2^7 \frac{dx}{x} = \frac{1}{2}\left[0.5 + 2(0.3333 + 0.25 + 0.2 + 0.166) + 0.1428\right]$$

$$= \frac{1}{2}\left[0.5 + 1.8986 + 0.1428\right]$$

$$= 1.2707$$

2. Evaluate $\int_0^2 \frac{dx}{16+x^2}$ using trapezoidal rule to 6 sub intervals ~~to approximate~~

$x_0 = 0 \quad x_0 + nh = 2 \quad n = 6$

$0 + 6h = 2$

$h = \frac{1}{3}$

$\frac{4}{3} + \frac{1}{3}$

x	0	$\frac{1}{3}$	$2/3$	1	$\frac{4}{3}$	$5/3$	2
y	~~0.019~~ 0.0625	0.0620	0.0608	0.0588	0.0562	0.0532	0.05

$$\int_{x_0}^{x_0+nh} y\,dx = \frac{h}{2}\left[y_0 + 2(y_1 + y_2 + y_3 + \cdots y_{n-1}) + y_n\right]$$

$$= \frac{1}{6}\left[0.0625 + 2(0.0620 + 0.0608 + 0.0588 + 0.0562 + 0.0532) + 0.05\right]$$

$$= 0.1166$$

Find the solution using trapezoidal rule

x	1.4	1.6	1.8	2.0	2.2
y	4.0552	4.9530	6.0436	7.3891	9.0250

$x_0 = 1.4 \qquad h = 0.2$

$\cancel{(x_0 + nh)}$

$= \frac{0.2}{2}\left[4.0552 + 2(4.9530 + 6.0436 + 7.3891) + 9.0250\right]$

$= \cancel{4.98552}\ 4.98516$

$= \underline{\underline{5}}$

Evaluate $I = \int_0^1 \frac{1}{1+x^2}\,dx$ by simpson's $\frac{1}{3}$rd rule by considering 6 sub intervals of the interval $[0\ 1]$. hence find an approximate value of π

$x_0 = 0 \qquad x_0 + nh = 1 \qquad n = 6$

$h = \frac{1}{6}$

x	0	$\frac{1}{6}$	$\frac{2}{3}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{5}{6}$	1
y	1	0.9729	$\cancel{0.3076}$ 0.9	0.8	0.6923	0.5902	0.5

$\frac{1}{6} + \frac{1}{6}$

$\frac{1}{3} + \frac{1}{6}$

$\frac{2}{6} + \frac{1}{6}$ $3/6\ 2$

$\frac{1}{2} + \frac{1}{6}$

$\frac{3}{6} + \frac{1}{6}$

$\frac{4}{6}$

$\frac{4}{6} + \frac{1}{6}$ $\frac{2}{3} + \frac{1}{6}$

By simpson's $\frac{1}{3}$rd rule

$$\int_0^1 \frac{1}{1+x^2}\,dx = \frac{h}{3}\left[(y_0+y_6)+4(y_1+y_3+y_5)+2(y_2+y_4)\right]$$

$$= \frac{1}{18}\left[(1+0.5)+4(0.973+0.8+0.502)+2(0.9+0.6923)\right]$$

$$= 0.7854$$

$$\int_0^1 \frac{1}{1+x^2}\,dx = 0.7854$$

$$\tan^{-1}x \Big|_0^1 = 0.7854$$

$$\tan^{-1}(1) - \tan^{-1}(0) = 0.7854$$

$$\frac{\pi}{4} - 0 = 0.7854$$

$$\pi = 4 \times 0.7854$$

$$\pi = 3.1416$$

Evaluate integral $\int_0^{\pi/2} \cos x\, dx$ by applying simpsons $\frac{1}{3}$rd rule by taking 11 ordinates

$x_0 = 0$ $\quad x_0 + nh = \frac{\pi}{2}$ $\quad n+1 = 11$ (ordinates)

$n = 10$

$0 + 10h = \frac{\pi}{2}$

$h = \frac{\pi}{20}$ $\quad y = \cos x$

$h = 9^\circ$

x	0	9	18	27	36	45
y	1	0.9877 (y_1)	0.9510 (y_2)	0.8910 (y_3)	0.8090 (y_4)	0.7071 (y_5)

x	54	63	72	81	90
y	0.5878 (y_6)	0.4540 (y_7)	0.3090 (y_8)	0.1564 (y_9)	0 (y_{10})

By simpson's $\frac{1}{3}$rd rule

$$\int_0^{\pi/2} \cos x\, dx = \frac{h}{3}\Big[(y_0 + y_{11}) + 4(y_1 + y_3 + y_5 + y_7 + y_9 \cancel{+ y_{11}}) + 2(y_2 + y_4 + y_6 + y_8 + y_{10})\Big]$$

$$= \frac{9}{3}\Big((1+0) + 4(0.9877 + \cancel{0.9510 + 0.8910 +} 0.8910 + 0.7071 + 0.4540 + 0.1564) + 2(0.9510 + 0.8090 + 0.5878 + 0.3090 + 0)$$

$= 1$ //

19.0984

Evaluate $I = \int_0^5 \frac{1}{4x+5}\,dx$ by simpsons $\frac{1}{3}$rd rule by considering 10 sub intervals of the interval hence find an approximate value of log 5

$x_0 = 0 \qquad x_0 + nh = 5. \qquad n = 10$

$0 + 10h = 5$

$h = \frac{1}{2}$

x	0	0.5	1	1.5	2	2.5
y	0.5	0.1428	0.1111	0.0909	0.0796	0.0666
		y_1	y_2	y_3	y_4	y_5

3	3.5	4	4.5	5
0.0588	0.0526	0.0476	0.0434	0.04
y_6	y_7	y_8	y_9	y_{10}

By simpson's $\frac{1}{3}$rd rule

$$= \frac{0.5}{3}\Big[(0.2 + 0.04) + 4(0.1428 + 0.0909 + 0.0666 + 0.0526 + 0.0434) + 2(0.1111 + 0.0796 + 0.0588 + 0.0476)\Big]$$

$= 0.4023$

$$\int_0^5 \frac{1}{4x+5}\,dx = 0.4023 \qquad \frac{d}{dx}\log(4x+5) = \frac{1}{(4x+5)}\times 4$$

$$\frac{1}{4}\int_0^5 \frac{4}{4x+5}\,dx = 0.4023$$

$$\frac{1}{4}\,\log(4x+5)\Big|_0^5 = 0.4023$$

$$\log(4\times 20+5) - \log(5) = 4\times 0.4023$$

$$\log\left(\frac{25}{5}\right) = 1.6092$$

$$\log 5 = 1.6092$$