IV - Integral Calculus

Multiple integrals :⇒
Evaluation of double and triple integrals
Evaluation of double integrals by change of order of integration, changing into polar Coordinates. Applications to find area and volume by double integral problems
Beta and Gamma funⁿ :⇒
Defⁿ, properties, Relⁿ betⁿ Beta & Gamma funⁿs, problems.

Double and Triple integrals
Double integrals → $\iint f(x,y)\,dx\cdot dy$

Triple integrals → $\iiint f(x,y,z)\,dx.dy.dz$

i) $\iint (x+y)\,dx\cdot dy$

$\Rightarrow \int \left[\frac{x^2}{2}+xy\right] dy$

$\Rightarrow \frac{x^2}{2}\cdot y + \frac{x\cdot y^2}{2}$

$\Rightarrow \frac{xy}{2}[x+y]$

ii) $\int_{x=0}^{1}\int_{y=0}^{2}\int_{z=0}^{3} xyz\,dz\cdot dy\cdot dx$

$\Rightarrow \int_{x=0}^{1}\int_{y=0}^{2} xy\left[\frac{z^2}{2}\right]_0^3 dy\cdot dx = \int_{x=0}^{1}\int_{y=0}^{2} xy\left[\frac{9}{2}-0\right] dy\,dx$

$\Rightarrow \frac{9}{2}\int_{x=0}^{1}\int_{y=0}^{2} xy\cdot dy\,dx \Rightarrow \frac{9}{2}\int_{x=0}^{1} x\left[\frac{y^2}{2}\right]_0^2 dx$

$\Rightarrow \frac{9}{2}\int_{x=0}^{1} x\cdot\left[\frac{4}{2}-0\right] dx$

$\Rightarrow \frac{9\times 2}{2}\int_{x=0}^{1} x\,dx$

$\Rightarrow 9\int_{x=0}^{1} x\,dx \Rightarrow 9\left[\frac{x^2}{2}\right]_0^1 \Rightarrow 9\left[\frac{1}{2}\right] \Rightarrow \frac{9}{2}$

3) Evaluate $\int_0^1\int_x^{\sqrt{x}} (x^2+y^2)\,dy\,dx$

$\Rightarrow \int_0^1 \left[\frac{x^3}{3}+xy^2\right]_x^{\sqrt{x}} dx$

$\Rightarrow \int_0^1 \left[x^2\cdot y + \frac{y^3}{3}\right]_x^{\sqrt{x}} dx$

$\Rightarrow \int_0^1 \left[x^2\cdot\sqrt{x} + \frac{(\sqrt{x})^3}{3} - x^3 - \frac{x^3}{3}\right] dx$

$\Rightarrow \int_0^1 x^{5/2} + \frac{x^{3/2}}{3} - x^3 - \frac{x^3}{3}\,dx$

$\Rightarrow \int_{x=0}^{1} x^{5/2} + \frac{x^{3/2}}{3} - \frac{4x^3}{3}\,dx$

$\Rightarrow \left[\frac{x^{7/2}}{7/2} + \frac{1}{3}\frac{x^{5/2}}{5/2} - \frac{4}{3}\frac{x^4}{4}\right]_0^1$

$\Rightarrow \left[\frac{2}{7}x^{7/2} + \frac{2}{15}x^{5/2} - \frac{1}{3}x^4\right]_0^1$

$\Rightarrow \left[\frac{2}{7}(1) + \frac{2}{15}(1) - \frac{1}{3}(1) - 0\right]$

$\Rightarrow \frac{30+14-35}{105} = \frac{44-35}{105} = \frac{9}{105} = \frac{3}{35}$

4> Evaluate $\int_{-c}^{c}\int_{-b}^{b}\int_{-a}^{a} (x^2+y^2+z^2)\, dz\cdot dy\cdot dx$

Sol^n :=> $I = \int_{x=-c}^{c}\int_{y=-b}^{b}\int_{z=-a}^{a} (x^2+y^2+z^2)\, dz\cdot dy\cdot dx$

$I = \int_{x=-c}^{c}\int_{y=-b}^{b} \left[x^2 z + y^2 z + \frac{z^3}{3}\right]_{-a}^{a} dy\cdot dx$

$I = \int_{x=-c}^{c}\int_{y=-b}^{b} \left[x^2[a+a] + y^2[a+a] + \frac{[a^3+a^3]}{3}\right] dy\, dx$

$I = \int_{x=-c}^{c}\int_{y=-b}^{b} \left[2ax^2 + 2ay^2 + \frac{2a^3}{3}\right] dy\, dx$

$I = \int_{x=-c}^{c} \left[2ax^2 y + \frac{2ay^3}{3} + \frac{2a^3}{3}y\right]_{-b}^{b} dx$

$I = \int_{x=-c}^{c} \left[2ax^2[b+b] + 2a\frac{[b^3+b^3]}{3} + \frac{2a^3}{3}[b+b]\right] dx$

$I = \int_{x=-c}^{c} \left[4abx^2 + \frac{4ab^3}{3} + \frac{4a^3b}{3}\right] dx$

$I = \left[4ab\frac{x^3}{3} + \frac{4ab^3}{3}x + \frac{4a^3b\cdot x}{3}\right]_{-c}^{c}$

$I = \left[4ab\frac{[c^3+c^3]}{3} + \frac{4ab^3[c+c]}{3} + \frac{4a^3b[c+c]}{3}\right]$

$I = \frac{8abc^3}{3} + \frac{8ab^3c}{3} + \frac{8a^3bc}{3}$

$I = \frac{8abc}{3}[a^2+b^2+c^2]$ //

5> Evaluate $\int_{-1}^{1}\int_{0}^{z}\int_{x-z}^{x+z} (x+y+z)\, dy\cdot dx\cdot dz$

Sol^n => $I = \int_{z=-1}^{1}\int_{x=0}^{z}\int_{y=x-z}^{x+z} (x+y+z)\, dy\cdot dx\cdot dz$

$\frac{x^2+z^2+2xz - x^2 - z^2 + 2xz}{}$
$4xz$

$I = \int_{z=-1}^{1}\int_{x=0}^{z} \left[xy + \frac{y^2}{2} + zy\right]_{x-z}^{x+z} dx\cdot dz$

$I = \int_{z=-1}^{1}\int_{x=0}^{z} \Big[x[x+z-x+z] + \frac{[(x+z)^2-(x-z)^2]}{2} +$

$z[(x+z-x+z)]\Big]\, dx\cdot dz$

$I = \int_{z=-1}^{1}\int_{x=0}^{z} \left[2xz + \frac{4xz}{2} + 2z^2\right] dx\cdot dz$

$I = \int_{z=-1}^{1} \left[\frac{2x^2}{2}\cdot z + \frac{2x^2}{2}\cdot z + 2z^2 x\right]_0^z dz$

$I = \int_{z=-1}^{1} \left[x^2 z + x^2 z + 2z^2 x\right]_0^z dz$

$I = \int_{z=-1}^{1} \left[z^3 + z^3 + 2z^3 - 0\right] dz$

$I = \left[\frac{4z^4}{4}\right]_{-1}^{1} = \left[z^4\right]_{-1}^{1} \Rightarrow (1)^4 - (-1)^4$

$I = 1 - 1 = 0$

6> Evaluate $\int_{0}^{1}\int_{0}^{\sqrt{1-x^2}}\int_{0}^{\sqrt{1-x^2-y^2}} xyz\, dz\cdot dy\cdot dx$

Sol^n :=> $I = \int_{x=0}^{1}\int_{y=0}^{\sqrt{1-x^2}}\int_{z=0}^{\sqrt{1-x^2-y^2}} xyz\, dz\cdot dy\cdot dx$

$I = \int_{x=0}^{1}\int_{y=0}^{\sqrt{1-x^2}} \left[xy\left[\frac{z^2}{2}\right]_0^{\sqrt{1-x^2-y^2}}\right] dy\, dx$

$I = \int_{x=0}^{1}\int_{y=0}^{\sqrt{1-x^2}} \left[xy\frac{(1-x^2-y^2)}{2} - 0\right] dy\, dx$

$I = \int_{x=0}^{1}\int_{y=0}^{\sqrt{1-x^2}} \frac{xy - x^3y - xy^3}{2}\, dy\, dx$

$I = \frac{1}{2}\int_{x=0}^{1} \left[x\frac{y^2}{2} - x^3\frac{y^2}{2} - x\frac{y^4}{4}\right]_0^{\sqrt{1-x^2}} dx$

$$I=\frac{1}{2}\int_{x=0}^{1}\left[\frac{x(1-x^2)}{2}-\frac{x^3(1-x^2)}{2}-\frac{x}{4}(1+x^2)^2\right]dx$$

$$I=\frac{1}{2}\int_{x=0}^{1}\left[\frac{x}{2}-\frac{x^3}{2}-\frac{x^3}{2}+\frac{x^5}{2}-\frac{x}{4}+\frac{x^2}{4}+\frac{2x^2}{4}\right]dx$$

$$I=\frac{1}{2}\int_{x=0}^{1}\left[\frac{1}{4}\left[2x-2x^3-2x^3+2x^5-x-x^2+2x^2\right]dx\right.$$

$$I=\frac{1}{2}\int_{x=0}^{1}\frac{1}{4}\left[2x-4x^3+2x^5+x^2\right]dx$$

$$I=\frac{1}{2}\int_{x=0}^{1}\frac{1}{4}\left[x+x^2-4x^3+2x^5\right]dx$$

$$I=\frac{1}{2}\cdot\frac{1}{4}\left[\frac{x^2}{2}+\frac{x^3}{3}-\frac{\cancel{4}x^4}{\cancel{4}}+\frac{\cancel{2}x^6}{\cancel{6}\,3}\right]_0^1$$

$$I=\frac{1}{8}\left[\frac{1}{2}+\frac{1}{3}-1+\frac{1}{3}\right]$$

$$I=\frac{1}{8}\left[\frac{3+2-6+2}{6}\right]=\frac{1}{8}\left[\frac{1}{6}\right]=\frac{1}{48}.$$

Evaluation of a double integral by Changing the order of integration

1) Evaluate $\int_0^1\int_x^{\sqrt{x}} xy\,dy\,dx$ by changing the order of integration

Sol^n :=> $I=\int_{x=0}^{1}\int_{y=x}^{\sqrt{x}} xy\,dy\,dx$

We need to first identify the region of integration R bounded by the curves $y=x$, $y=\sqrt{x}$ between the lines $x=0$, $x=1$.

$y=x$
$y=\sqrt{x} \Rightarrow y^2=x$

we shall find the points of intersection of $y=x$ and $y=\sqrt{x}$ by equating their RHS.
i.e $x=\sqrt{x} \Rightarrow x^2=x$, or $x(x-1)=0$, $x=0$, $x=1$
This will give us $y=0$, $y=1$ hence the points of intersection are $(0,0)$ and $(1,1)$.
Further w.k.t $y=x$ is a straight line passing through the origin making an angle 45° with the x axis & $y=\sqrt{x}$ or $y^2=x$ is a parabola symmetrical about the x-axis. The befitting figure indicating R is given.

On changing the order of integration we must have Constant limits for y and variable limits for x. From the figure we observe that y varies from 0 to 1 and x varies from y^2 ($\because y=\sqrt{x}$) to y

Thus we have on changing the order of integration

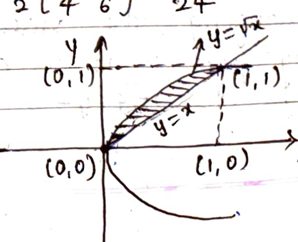
$$I=\int_{y=0}^{1}\int_{x=y^2}^{y} xy\,dx\,dy$$

$$=\int_{y=0}^{1} y\left[\frac{x^2}{2}\right]_{x=y^2}^{y}dy=\frac{1}{2}\int_{y=0}^{1} y(y^2-y^4)\,dy$$

$$I=\frac{1}{2}\int_0^1(y^3-y^5)\,dy$$

$$=\frac{1}{2}\left[\frac{y^4}{4}-\frac{y^6}{6}\right]_0^1=\frac{1}{2}\left[\frac{1}{4}-\frac{1}{6}\right]=\frac{1}{24}$$

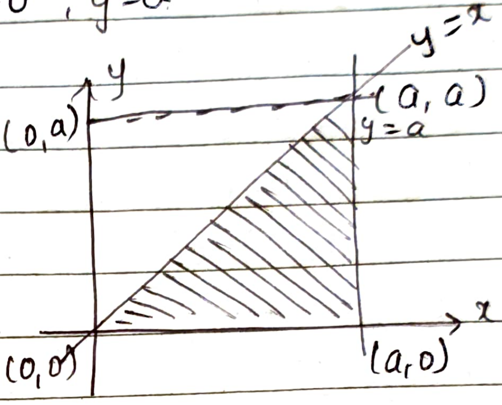
$y=0\to 1$
$y=\sqrt{x}\to x$
$x=y$
$x=y^2$



3) $\int_0^a \int_y^a \frac{x}{x^2+y^2} \, dx \cdot dy$ by change of order.

Sol:- $\Rightarrow$ $I = \int_{y=0}^{a} \int_{x=y}^{a} \frac{x}{x^2+y^2} \, dx \cdot dy$

$x=y$, $x=a$, $y=0$, $y=a$



$x = 0$ to a

$y = 0$ to x

$I = \int_{x=0}^{a} \int_{y=0}^{x} x \cdot \frac{1}{x^2+y^2} \, dy \, dx$

$I = \int_{x=0}^{a} \cancel{x} \cdot \frac{1}{\cancel{x}} \left[\tan^{-1}\left(\frac{y}{x}\right) \right]_{y=0}^{x} dx$

$I = \int_{x=0}^{a} \left[\tan^{-1}(x/x) - \tan^{-1}(0/x) \right] dx$

$I = \int_{x=0}^{a} \left[\tan^{-1}(1) - \tan^{-1}(0) \right] dx$

$I = \int_{x=0}^{a} \frac{\pi}{4} \, dx = \frac{\pi}{4} [x]_0^a$

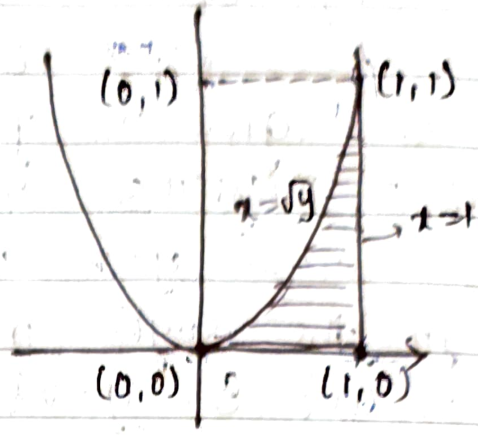
$I = \frac{\pi}{4} \times (a-0)$

$I = \frac{\pi a}{4}$ //

2> Change the order of the integration and hence evaluate $\int_{y=0}^{1}\int_{\sqrt{y}}^{1} dx\, dy$

Sol.ⁿ :⇒ let

$$I = \int_{y=0}^{1}\int_{x=\sqrt{y}}^{1} dx.dy$$



On changing the order Integrⁿ

$$I = \int_{x=0}^{1}\int_{y=0}^{x^2} dy\, dx \quad (x=\sqrt{y} \Rightarrow x^2 = y)$$

$$= \int_{x=0}^{1} [y]_0^{x^2} dx$$

$$= \int_{x=0}^{1} x^2 dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3}$$

$y = \sqrt{4-x^2}$
$y^2 = 4 - x^2$
$y^2 + x^2 = 4$

3> Evaluate $\int_{-2}^{2}\int_{0}^{\sqrt{4-x^2}} (2-x)\, dy\, dx$ by changing the order of integration

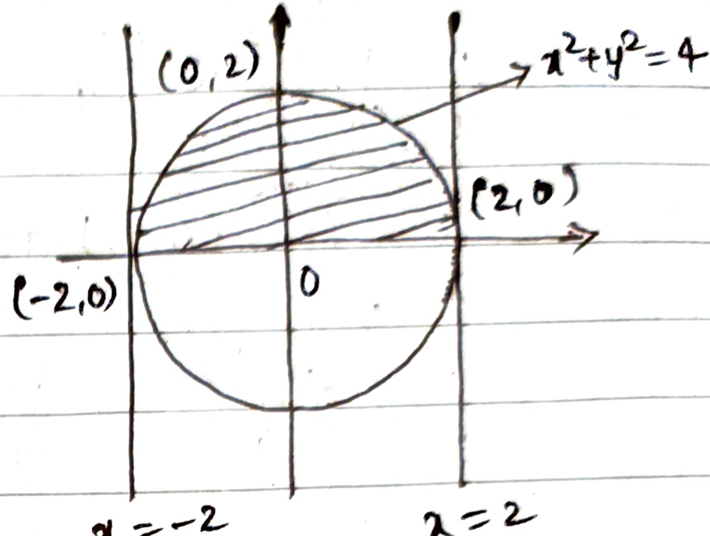
⇒ we have

$$I = \int_{x=-2}^{2}\int_{y=0}^{\sqrt{4-x^2}} (2-x)\, dy\, dx$$

Here, $y = \sqrt{4-x^2}$ or $x^2+y^2 = 4$
This is a circle with the centre origin and radius 2, $y=0$ to $\sqrt{4-x^2}$ is the upper half of the circle being bounded by the lines $x=-2$, & 2
On changing the order we must have y varying from 0 to 2 and x from $-\sqrt{4-y^2}$ to $\sqrt{4-y^2}$

$$\frac{d}{dx}\int\sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)+c$$

$$I=\int_{y=0}^{2}\int_{x=-\sqrt{4-y^2}}^{\sqrt{4-y^2}}(2-x)\,dx\,dy$$



$$I=\int_{y=0}^{2}\left[2x-\frac{x^2}{2}\right]_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}}dy$$

$$I=\int_{y=0}^{2}\left[2\cdot 2\sqrt{4-y^2}-0\right]dy$$

$$I=4\int_{y=0}^{2}\sqrt{2^2-y^2}\,dy$$

$$I=4\left[\frac{y\sqrt{4-y^2}}{2}+\frac{2^2}{2}\sin^{-1}\frac{y}{2}\right]_0^2$$

$$=4(0+2\sin^{-1}1)=8\cdot\frac{\pi}{2}=4\pi$$

4) Change the order of integration and hence Evaluate $\int_0^{4a}\int_{x^2/4a}^{2\sqrt{ax}} xy \cdot dy \cdot dx$.

Sol^n:- we have $I = \int_{x=0}^{4a}\int_{y=x^2/4a}^{2\sqrt{ax}} xy \cdot dy\, dx$

$\frac{x^2}{4a} = 2\sqrt{ax}$.

$x^4 = 64a^3x$

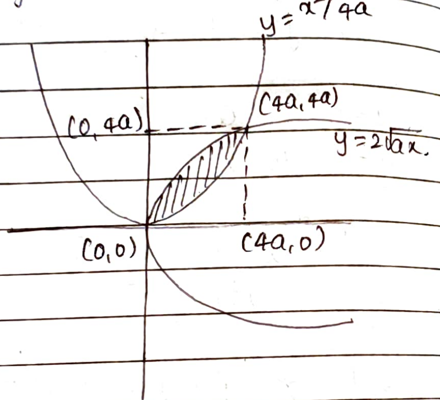
$x^4 - 64a^3x = 0$

$x(x^3 - 64a^3) = 0$

$x = 0,\ x = 4a$

from $y = \frac{x^2}{4a}$

$y = 0$ & $y = 4a$.



Hence the points of intersection of the parabolae $y = \frac{x^2}{4a}$ and $y = 2\sqrt{ax}$ are $(0,0)$ & $(4a, 4a)$

On changing the order of integration, we have y varying from 0 to $4a$ and x varying from $\frac{y^2}{4a}$ to $2\sqrt{ay}$

Now, $I = \int_{y=0}^{4a}\int_{x=y^2/4a}^{2\sqrt{ay}} xy\, dx \cdot dy$

$I = \int_{y=0}^{4a} \left[\frac{x^2}{2}\right]_{y^2/4a}^{2\sqrt{ay}} \cdot y\, dy$

$I = \int_{y=0}^{4a} \frac{y}{2}\left[(2\sqrt{ay})^2 - \left(\frac{y^2}{4a}\right)^2\right] dy$

$I = \int_{y=0}^{4a} \frac{y}{2}\left[4ay - \frac{y^4}{16a^2}\right] dy$

$I = \int_{y=0}^{4a} \frac{4ay^2}{2} - \frac{y^5}{32a^2}\, dy$

$I = \frac{4a}{2}\left[\frac{y^3}{3}\right]_0^{4a} - \frac{1}{32a^2}\left[\frac{y^6}{6}\right]_0^{4a}$

$I = \frac{4a}{2}\left[\frac{64a^3 - 0}{3}\right] - \frac{1}{32a^2}\left[\frac{64 \times 64 a^6}{6}\right]$

$I = \frac{4a(64a^3)}{6} - \frac{64 \times 64 a^6}{192 a^2}$

$I = \frac{1}{2}\left[\frac{256a^4}{3} - \frac{4096 a^4}{96}\right]$

$I = \frac{1}{2}\left[\frac{256a^4}{3} - \frac{128a^4}{3}\right]$

$I = \frac{1}{2}\left[\frac{128a^4}{3}\right]$

$I = \frac{64a^4}{3}$ //

H/w:

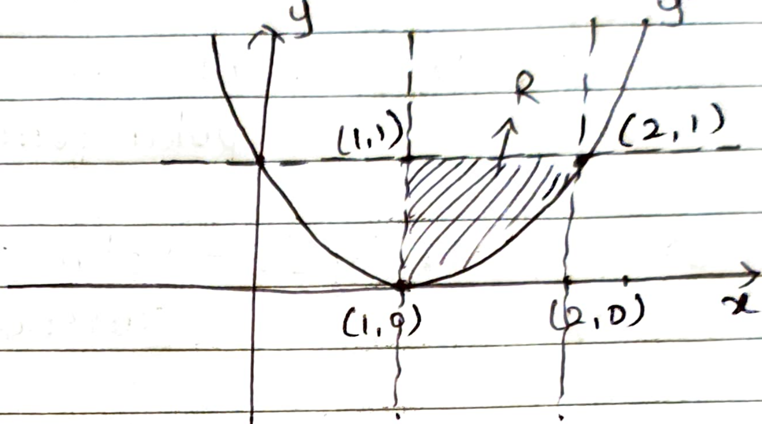
Change the integral $\int_{-a}^{a}\int_{0}^{\sqrt{a^2-x^2}} \sqrt{x^2+y^2}\, dy\, dx$ into polars and hence evaluate the same.

$y^2 \cdot y^{1/2} = y^{2+1/2}$
$= y^{5/2}$

1) Evaluate $\int_1^2 \int_1^{x^2} (x^2+y^2)\, dy\, dx$ by COI

Sol^n :⇒ Given. $x=1$, $x=2$
$y=1$, $y=x^2$

By change of order from the figure.



$y=0$, $y=1$
$x=1$, $x=\sqrt{y}$

Now.

$$I = \int_{y=0}^{1} \int_{x=1}^{\sqrt{y}} (x^2+y^2) \cdot dx \cdot dy$$

$$I = \int_{y=0}^{1} \left[\frac{x^3}{3}\right]_1^{\sqrt{y}} + y^2[x]_1^{\sqrt{y}} \cdot dy$$

$$I = \int_{y=0}^{1} \left[\frac{(\sqrt{y})^3 - 1}{3}\right] + y^2[\sqrt{y}-1] \cdot dy$$

$$I = \int_{y=0}^{1} \frac{y^{3/2}}{3} - \frac{1}{3} + y^{5/2} - y^2 \quad dy$$

$$I = \left[\frac{1}{3} \times \frac{y^{5/2}}{5/2} - \frac{1}{3}y + \frac{y^{7/2}}{7/2} - \frac{y^3}{3}\right]_0^1$$

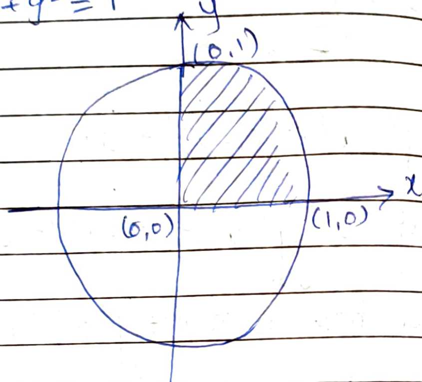
$$I = \frac{2}{15}[1-0] - \frac{1}{3}[1-0] + \frac{2}{7}[1-0] - \frac{1}{3}[1-0]$$

$$I = \frac{2}{15} - \frac{1}{3} + \frac{2}{7} - \frac{1}{3} = \frac{14-35+30-35}{105}$$

$$I = \frac{44-70}{105} = \frac{-26}{105}$$

* Evaluate by changing the order of integration. $\int_0^1 \int_0^{\sqrt{1-x^2}} y^2 \cdot dy \cdot dx$

Sol^n :->
$x = 0, \ x = 1$
$y = 0, \ y = \sqrt{1-x^2}$
$x^2 + y^2 = 1$



$$I = \int_{y=0}^{1} \int_{x=0}^{\sqrt{1-y^2}} y^2 \cdot dx \cdot dy$$

$$I = \int_0^1 y^2 \left[x \right]_0^{\sqrt{1-y^2}} \cdot dy$$

$$I = \int_0^1 y^2 \left[\sqrt{1-y^2} - 0 \right] dy$$

$$I = \int_0^1 y^2 \left(\sqrt{1-y^2} \right) \cdot dy$$

put $y = \sin\theta$
$dy = \cos\theta \cdot d\theta$
θ varies from 0 to $\pi/2$

$$I = \int_0^{\pi/2} \sin^2\theta \cdot \sqrt{1 - \sin^2\theta} \cdot \cos\theta \cdot d\theta$$

$$I = \int_0^{\pi/2} \sin^2\theta \ \cancel{\sqrt{\cos^2\theta}} \cdot \cos\theta \ d\theta$$

$$I = \int_0^{\pi/2} \sin^2\theta \cdot \cos^2\theta \cdot d\theta$$

~~By the def^n Beta fun^n we have to~~

By Reduction formula
$$\int_0^{\pi/2} \sin^m x \cdot \cos^n x \cdot dx = \frac{[(m-1)(m-3) \ldots][(n-1)(n-3) \ldots] \times k}{(m+n)(m+n-2) \ldots}$$

$$\int_0^{\pi/2} \sin^2\theta \cdot \cos^2\theta \ d\theta = \frac{(2-1)(2-1)}{(2+2)(2+2-2)} \times \frac{\pi}{2}$$

$$\int_0^{\pi/2} \sin^2\theta \cdot \cos^2\theta \cdot d\theta = \frac{1}{4 \times 2} \times \frac{\pi}{2}$$

$$= \frac{\pi}{16} \ //$$

Evaluation by changing into polars

1) Evaluate $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\, dx.dy$ by changing to polar Coordinates.

Sol^n :- In polar we have $x = r\cos\theta$, $y = r\sin\theta$

$\therefore x^2+y^2 = r^2$ and $dx.dy = r.dr.d\theta$

Since x, y varies from 0 to ∞

r also varies from 0 to ∞.

In the first quadrant θ varies from 0 to $\pi/2$

Thus $I = \int_{\theta=0}^{\pi/2} \int_{r=0}^{\infty} e^{-r^2}.r.dr\, d\theta$

put $r^2 = t$ $\quad \therefore r.dr = \frac{dt}{2}$

t also varies from 0 to ∞

$I = \int_{\theta=0}^{\pi/2} \int_{t=0}^{\infty} e^{-t}.\frac{dt}{2}\, d\theta$

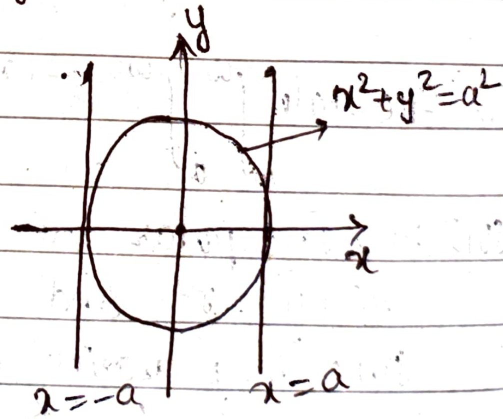
$$= \frac{1}{2} \int_{\theta=0}^{\pi/2} \left[-e^{-t}\right]_{t=0}^{\infty} d\theta$$

$$= -\frac{1}{2} \int_{\theta=0}^{\pi/2} (0-1)\, d\theta$$

$$I = \frac{1}{2} [\theta]_0^{\pi/2} = \frac{\pi}{4}$$

2> Change the integral $\int_{-a}^{a} \int_{0}^{\sqrt{a^2-x^2}} \sqrt{x^2+y^2}\, dy\, dx$ into polar and hence evaluate the same

Sol^n :=> The region of integration is as shown in the figure



Clearly θ varies from 0 to π

If $x = r\cos\theta$, $y = r\sin\theta$, $x^2+y^2=r^2$

i.e $a^2 = r^2 \Rightarrow r = a$

$\therefore$ r varies from 0 to a

Also $dx\,dy = r.dr.d\theta$

$$\therefore I = \int_{\theta=0}^{\pi} \int_{r=0}^{a} r.r.dr.d\theta$$

$$I = \int_{\theta=0}^{\pi} \left[\frac{r^3}{3}\right]_0^a d\theta$$

$$= \frac{a^3}{3} [\theta]_0^{\pi} = \frac{a^3}{3}(\pi - 0) = \frac{\pi a^3}{3}$$

2> Evaluate $\int_0^1 \int_0^{\sqrt{1-y^2}} (x^2+y^2)\, dx\, dy$ by changing into polar coordinates

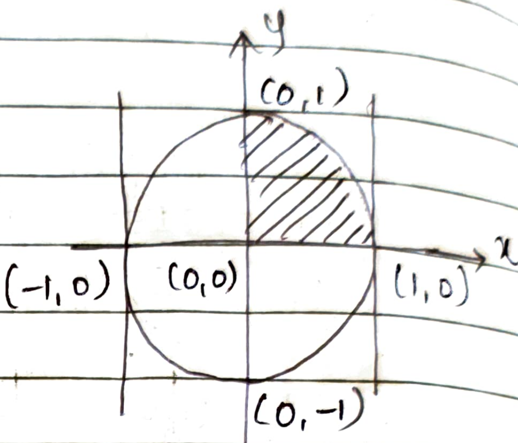
Sol^n:- $x = r\cos\theta$, $y = r\sin\theta$, $x^2+y^2 = r^2$

$dy \cdot dx = r \cdot dr \cdot d\theta$

$y = 0$, $y = 1$

$x = 0$, $x = \sqrt{1-y^2}$

$x^2+y^2 = 1$



$r \Rightarrow 0$ to 1

$\theta \Rightarrow 0$ to $\pi/2$

$$I = \int_0^{\pi/2} \int_0^1 (r^2\cos^2\theta + r^2\sin^2\theta)\, r \cdot dr \cdot d\theta$$

$$I = \int_0^{\pi/2} \int_0^1 r^3 (\cos^2\theta + \sin^2\theta)\, dr \cdot d\theta$$

$$I = \int_0^{\pi/2} \left[\frac{r^4}{4}\right]_0^1 d\theta$$

$$I = \int_0^{\pi/2} \left[\frac{1}{4} - 0\right] d\theta$$

$$I = \frac{1}{4} \int_0^{\pi/2} d\theta$$

$$I = \frac{1}{4} [\theta]_0^{\pi/2}$$

$$I = \frac{1}{4} \times \frac{\pi}{2}$$

$$\boxed{I = \pi/8}$$

Applications to find Area and volume.

1> $\iint_R dx \cdot dy$ = Area of the region R in the cartesian form.

2> $\iint_R r \cdot dr \cdot d\theta$ = Area of the region R in the polar form

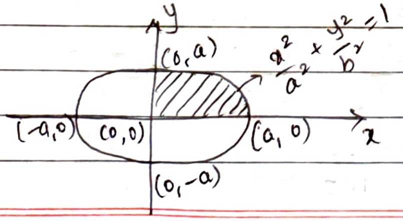
3> $\iiint_V dx \cdot dy \cdot dz$ = Volume of the solid in the cartesian form

4> $\iint_A 2\pi r^2 \cdot \sin\theta \cdot dr \cdot d\theta$ = Volume of a solid obtained by the revolution of a curve enclosing an area A about the initial line in the polar form.

Problems :-

1> Find the area of ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ by double integration.

Soln :⇒ Area $(A) = \iint_R dx \cdot dy$.



From the figure we can write

$A = 4A_1$ where A_1 is the area in the first quadrant and with reference to the same figure we have.

$$A = 4A_1$$

$$= 4 \int_{x=0}^{a} \int_{y=0}^{(b/a)\sqrt{a^2-x^2}} dy\, dx$$

$$= 4 \int_{x=0}^{a} [y]_0^{(b/a)\sqrt{a^2-x^2}} dx$$

$$= 4 \int_{x=0}^{a} \frac{b}{a} \sqrt{a^2-x^2} \cdot dx$$

$$= \frac{4b}{a} \left[\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2} \sin^{-1}\frac{x}{a} \right]_{x=0}^{a}$$

$$= \frac{4b}{a} \left[0 + \frac{a^2}{2} (\sin^{-1} 1 - \sin^{-1} 0) \right]$$

$$= \frac{\cancel{4}^{2} b}{\cancel{a}} \cdot \frac{a^{\cancel{2}}}{\cancel{2}} \cdot \frac{\pi}{\cancel{2}}$$

$$A = \pi ab$$

Thus the required area $A = \pi ab$ sq. units

2> Find the volume of the tetrahedron bounded by the planes $x=0$, $y=0$, $z=0$ $x/a + y/b + z/c = 1$

Soln :⇒ $V = \iiint dx \cdot dy \cdot dz$

$$x/a + y/b + z/c = 1$$

$$\therefore z = c\left(1 - \frac{x}{a} - \frac{y}{b}\right)$$

If $z = 0$, then $\frac{x}{a} + \frac{y}{b} = 1$ $\therefore y = b\left(1 - \frac{x}{a}\right)$

If $z = 0$, $y = 0$ then $x = a$

$$\therefore V = \int_{x=0}^{a} \int_{y=0}^{b(1-x/a)} \int_{z=0}^{c(1-x/a-y/b)} dz \cdot dy \cdot dx.$$

On simple integration

$$V = \int_{x=0}^{a} \int_{y=0}^{b(1-x/a)} [z]_{0}^{c(1+x/a-y/b)} \cdot dy \cdot dx.$$

Beta and Gamma functions

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx \quad (m,n>0) \text{ --- } (1)$$

is called Beta function.

$$\Gamma(n) = \int_0^\infty e^{-x}\cdot x^{n-1}\cdot dx \quad (n>0) \text{ --- } (2)$$

is called Gamma function.

Alternative forms.

put $x = \sin^2\theta$.

$$dx = 2\sin\theta\cdot\cos\theta\cdot d\theta.$$

$x=0,\quad \sin^2\theta = 0 \quad \Rightarrow \theta = 0$

$x=1,\quad \sin^2\theta = 1 \quad \Rightarrow \theta = \pi/2$

$$\beta(m,n) = \int_{\theta=0}^{\pi/2} (\sin^2\theta)^{m-1}(\cos^2\theta)^{n-1}\cdot 2\sin\theta\cos\theta\cdot d\theta.$$

$$\beta(m,n) = 2\int_{\theta=0}^{\pi/2} (\sin\theta)^{2m-1} (\cos\theta)^{2n-1}\, d\theta$$

$$\beta(m,n) = 2\int_{\theta=0}^{\pi/2} \sin^{2m-1}\theta \cdot \cos^{2n-1}\theta \cdot d\theta \quad \text{—— (3)}$$

put $x = t^2$ in (2)

$dx = 2t \cdot dt$

$$\Gamma(n) = \int_{t=0}^{\infty} e^{-t^2} (t^2)^{n-1}\, 2t \cdot dt$$

$$\Gamma(n) = 2\int_{0}^{\infty} e^{-t^2}\, t^{2n-1}\, dt \quad \text{—— (4)}$$

Properties of Beta and Gamma fun^ns :=

1> $\beta(m,n) = \beta(n,m)$

2> $\Gamma(n+1) = n\,\Gamma(n)$ / $\Gamma(n) = (n-1)\,\Gamma(n-1)$

$\Gamma(n+1) = n!$ / $\Gamma(n) = (n-1)!$

3> $\Gamma(n+1) = n\Gamma(n)$ or $\Gamma(n) = \dfrac{\Gamma(n+1)}{n}$ and

this expression is used for finding $\Gamma(n)$ when n is a negative real number.

4> $\Gamma(1) = 1$

5> $\Gamma(1/2) = \sqrt{\pi}$

6> $\Gamma(1/4) \cdot \Gamma(3/4) = \pi\sqrt{2}$

7> $\displaystyle\int_0^{\pi/2} \sin^p\theta \cdot \cos^q\theta \cdot d\theta = \frac{1}{2}\,\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$

* Relationship between Beta & Gamma functions :-

$$\beta(m,n) = \frac{\Gamma(m)\cdot\Gamma(n)}{\Gamma(m+n)}.$$

proof :→ we have by the defn of β & Γ funcs.

$$\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta\cdot\cos^{2n-1}\theta\cdot d\theta \quad — (1).$$

$$\Gamma(n) = 2\int_0^{\infty} e^{-x^2}\cdot x^{2n-1}\cdot dx \quad — (2)$$

$$\Gamma(m) = 2\int_0^{\infty} e^{-y^2}\cdot y^{2m-1}\cdot dy \quad — (3).$$

$$\Gamma(m+n) = 2\int_0^{\infty} e^{-r^2}\cdot r^{2(m+n)-1}\cdot dr \quad — (4)$$

Now.

$$\Gamma(m)\cdot\Gamma(n) = 4\int_0^{\infty}\int_0^{\infty} e^{-(x^2+y^2)}\cdot x^{2n-1}\cdot y^{2m-1}\cdot dx\,dy \quad — (5)$$

Let us evaluate RHS by changing into polars.

putting $x = r\cos\theta,\ y = r\sin\theta$

we have $x^2+y^2 = r^2$

Also $dx\cdot dy = r\cdot dr\cdot d\theta$.

r varies from 0 to ∞, θ varies from 0 to $\pi/2$

we now have (5) in the form.

$$\Gamma(m)\cdot\Gamma(n) = 4\int_{r=0}^{\infty}\int_{\theta=0}^{\pi/2} e^{-r^2}(r\cos\theta)^{2n-1}(r\sin\theta)^{2m-1}\, r\cdot dr\cdot d\theta$$

$$= 4\int_{r=0}^{\infty}\int_{\theta=0}^{\pi/2} e^{-r^2}\cdot r^{2m+2n-1}\cdot\sin^{2m-1}\theta\cdot\cos^{2n-1}\theta\cdot dr\cdot d\theta.$$

$$= \left[2\int_{r=0}^{\infty} e^{-r^2} \cdot r^{2(m+n)-1} \cdot dr \right] \left[2\int_{\theta=0}^{\pi/2} \sin^{2m-1}\theta \cdot \cos^{2n-1}\theta \cdot d\theta \right]$$

$\therefore \Gamma(m) \cdot \Gamma(n) = \Gamma(m+n) \cdot \beta(m, n)$

{ by using ① & ④ }

Thus,

$$\beta(m,n) = \frac{\Gamma(m) \cdot \Gamma(n)}{\Gamma(m+n)}.$$

* prove that $\Gamma(1/2) = \sqrt{\pi}$

sol^n: w.k.t $\beta(m,n) = \dfrac{\Gamma(m) \cdot \Gamma(n)}{\Gamma(m+n)}$

put $m = 1/2$, and $n = 1/2$

$$\beta(1/2, 1/2) = \frac{\Gamma(1/2) \cdot \Gamma(1/2)}{\Gamma(1/2 + 1/2)}$$

$\beta(1/2, 1/2) = \{\Gamma(1/2)\}^2$ — ① $\{ \Gamma(1) = 1 \}$

Now consider,

$$\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cdot \cos^{2n-1}\theta \cdot d\theta$$

$$\beta(1/2, 1/2) = 2\int_0^{\pi/2} \sin^{2(1/2)-1}\theta \cdot \cos^{2(1/2)-1}\theta \cdot d\theta$$

$$= 2\int_0^{\pi/2} \sin^0\theta \cdot \cos^0\theta \cdot d\theta$$

$$= 2\int_0^{\pi/2} d\theta$$

$$= 2[\theta]_0^{\pi/2} = \cancel{2}\left[\frac{\pi}{\cancel{2}} - 0\right] = \pi$$

Now, we have from ①.

$\pi = \{\Gamma(1/2)\}^2 \Rightarrow \Gamma(1/2) = \sqrt{\pi}$ //

Alternate method :→

prove that $\Gamma(1/2) = \sqrt{\pi}$

sol^n: we have by the def^n

$$\Gamma(n) = 2\int_0^{\infty} e^{-x^2} \cdot x^{2n-1}\, dx$$

$$\Gamma(1/2) = 2\int_0^{\infty} e^{-x^2}\, dx = 2\int_0^{\infty} e^{-y^2} \cdot dy$$

Hence, $\{\Gamma(1/2)\}^2 = 4\int_0^{\infty}\int_0^{\infty} e^{-(x^2+y^2)}\, dx \cdot dy$

$$= 4\left[\frac{\pi}{4}\right]$$

$$\{\Gamma(1/2)\}^2 = \pi$$

$$\Gamma(1/2) = \sqrt{\pi} \quad //$$

Problems :→

1) show that $\int_0^{\pi/2} \frac{d\theta}{\sqrt{\sin\theta}} \times \int_0^{\pi/2} \sqrt{\sin\theta} \cdot d\theta = \pi$

sol^n :→ let $I_1 = \int_0^{\pi/2} \frac{d\theta}{\sqrt{\sin\theta}} = \int_0^{\pi/2} \sin^{-1/2}\theta \cdot d\theta$

$$= \int_0^{\pi/2} \sin^{-1/2}\theta \cdot \cos^0\theta \cdot d\theta$$

and $I_2 = \int_0^{\pi/2} \sqrt{\sin\theta} \cdot d\theta = \int_0^{\pi/2} \sin^{1/2}\theta \cdot d\theta$

$$= \int_0^{\pi/2} \sin^{1/2}\theta \cdot \cos^0\theta \cdot d\theta$$

we have

$$\int_0^{\pi/2} \sin^p\theta \cdot \cos^q\theta \cdot d\theta = \frac{1}{2}\, B\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$$

Hence $I_1 = \frac{1}{2}\beta\left(\frac{-1/2+1}{2}, \frac{0+1}{2}\right)$

$$I_1 = \frac{1}{2}\beta\left(\frac{1}{4}, \frac{1}{2}\right)$$

$$I_2 = \frac{1}{2}\beta\left(\frac{1/2+1}{2}, \frac{0+1}{2}\right)$$

$$I_2 = \frac{1}{2}\beta\left(\frac{3}{4}, \frac{1}{2}\right)$$

$$I_1 \times I_2 = \frac{1}{4}\beta\left(\frac{1}{4}, \frac{1}{2}\right)\cdot\beta\left(\frac{3}{4}, \frac{1}{2}\right)$$

$$= \frac{1}{4}\frac{\gamma(1/4)\cdot\gamma(1/2)}{\cancel{\gamma(3/4)}}\cdot\frac{\cancel{\gamma(3/4)}\cdot\gamma(1/2)}{\gamma(5/4)}$$

$$= \frac{1}{\cancel{4}}\,\cancel{\gamma(1/4)}\cdot\sqrt{\pi}\cdot\frac{\sqrt{\pi}}{1/4\cdot\cancel{\gamma(1/4)}}$$

$$= \sqrt{\pi}\cdot\sqrt{\pi}$$

$$I_1 \times I_2 = \pi$$

Hence proved.

2> Evaluate $\int_0^{\pi/2}\sqrt{\cot\theta}\cdot d\theta$. by Expressing interms of gamma fun^s

sol^n :- Let $I = \int_0^{\pi/2}\sqrt{\cot\theta}\cdot d\theta$

$$I = \int_0^{\pi/2}\frac{\sqrt{\cos\theta}}{\sqrt{\sin\theta}}\cdot d\theta$$

$$I = \int_0^{\pi/2}\sin^{-1/2}\theta\cdot\cos^{1/2}\theta\cdot d\theta.$$

Hence $I = \frac{1}{2}\beta\left(\frac{-1/2+1}{2}, \frac{1/2+1}{2}\right)$

$$I = \frac{1}{2}\beta\left(\frac{1}{4}, \frac{3}{4}\right)$$

$$= \frac{1}{2} \frac{\Gamma(1/4)\cdot\Gamma(3/4)}{\Gamma(1)}$$

$$= \frac{1}{2} \frac{\pi\sqrt{2}}{1} = \frac{\pi}{\sqrt{2}}$$ (since $\Gamma(1/4)\Gamma(3/4) = \pi\sqrt{2}$)

$$I = \frac{\pi}{\sqrt{2}}$$

3) $\int_0^1 x^{3/2}\cdot(1-x)^{1/2}\,dx$.

Let $I = \int_0^1 x^{3/2}\cdot(1-x)^{1/2}\,dx$.

we have $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$.

Taking $m-1 = \frac{3}{2}$, $n-1 = \frac{1}{2}$

we get $m = \frac{5}{2}$, $n = \frac{3}{2}$.

Hence, $I = \beta(5/2, 3/2) = \frac{\Gamma(5/2)\cdot\Gamma(3/2)}{\Gamma(4)}$

$$I = \frac{\Gamma(3/2+1)\cdot\Gamma(1/2+1)}{\Gamma(3+1)}$$

$$I = \frac{\frac{3}{2}\Gamma(3/2)\,\frac{1}{2}\Gamma(1/2)}{3!}$$

$$I = \frac{\frac{3}{4}\Gamma\left(\frac{1}{2}+1\right)\cdot\sqrt{\pi}}{6} = \frac{\frac{3}{8}\Gamma\left(\frac{1}{2}\right)\cdot\sqrt{\pi}}{6}$$

$$I = \frac{\cancel{3}\pi}{8\times\cancel{6}_{2}} = \frac{\pi}{16}$$ //