Vector space and Linear transformations.

Group :⇒ A non empty set equipped with one binary operation $*$ is called group if $*$ satisfies following postulates

1) Closure property :⇒ $\forall$ a and $b \in G \Rightarrow a * b \in G$
2) Associative property :⇒ $\forall a, b, c \in G \Rightarrow a*(b*c) = (a*b)*c$
3) Existance of identity :⇒
   There Exists an Element $e \in G$ such that
   $a * e = a = e * a$
   The Element e is called the identity
4) Existance of inverse :⇒
   Each Element of G possesses inverse i.e $\forall a \in G \Rightarrow \exists$ an Element $b \in G$ such that $a * b = e = b * a$
   Then b is called inverse of a and we write $b = a^{-1}$
   i.e $a * a^{-1} = e = a^{-1} * a$
5) A group with commutative property is known as abelian group or commutative group.
   i.e $\forall a, b \in G$, if $a * b = b * a$
   $\therefore$ $*$ is commutative on G

Ex: i) $(I, +)$ is an abelian group as $(I, +)$ is a group and also $\forall a, b \in I$
ii) $(Q_0, \cdot)$, $(R_0, \cdot)$, $(C_0, \cdot)$ are examples of abelian group

Rings :⇒
Suppose R is a non-empty set Equipped with two binary operation called addition and multiplication and denoted by '+' and '$\cdot$' respectively.

Commutative Ring with unity :⇒
Ring $(R, +, \cdot)$ is commutative ring with unity if $\forall a, b \in R \Rightarrow a \cdot b = b \cdot a$ and $\forall a \in R \ \exists \ 1 \in R$ such that $a \cdot 1 = a = 1 \cdot a$

Field :→

A commutative ring with unity in which every non-zero element possesses their multiplicative inverse is known as field.

Vector spaces :→

Let F be a field. A vector space over F is a non empty set V together with two operations (called addition and scalar multiplication) such that for each $u, v \in V$ there is a unique element $u+v \in V$ and for each $\alpha \in F$ and $u \in V$ there is a unique element $\alpha u \in V$ and it satisfies the following conditions

I i> $u+(v+w) = (u+v)+w$ for all $u, v, w \in V$ (Associativity)

ii> $(u+v) = (v+u)$ for all $u, v \in V$ (commutativity)

iii> There Exists an element $0 \in V$ such that $u+0 = u = 0+u$ for all $u \in V$ (Existance of identity element)

iv> for each $u \in V$, $\exists$ a unique element $-u \in V$ such that $u+(-u) = 0 = (-u)+u$ (Existance of additive inverse)

II There is an external composition in V over F called scalar multiplication i.e $\forall \alpha \in F$ and $u \in V \Rightarrow \alpha \cdot u \notin V$.
In other words V is closed with respect to scalar multi-plication

III The two compositions, i.e vector addition and scalar multi-plication satisfy the following postulates
$\forall a, b \in F$ and $v, w \in V$

i> $(a+b) \cdot v = a \cdot v + b \cdot v$

ii> $a \cdot (b \cdot v) = (ab) \cdot v$

iii> $a \cdot (v+w) = a \cdot v + a \cdot w$

iv> $1 \cdot (v) = v$.

Vectors in R^n :⇒

The set of all ordered Triples (a, b, c) of real numbers is called Euclidean 3 space and is denoted by R^3 and the set of all n-tuples of real no-s denoted by R^n is called Euclidean n-space.

Ex: The ordered pair $(2, -3)$ belongs to R^2, it is a 2-tuple of dimension two.
The ordered triple $(7, 3, 6)$ belongs to R^3, it is a 3-tuple of dimension three.

Subspaces :⇒

A non-empty subset W of a vector space $V(F)$ is said to form a subspace of V if W is also a vector space over F with the same addition and scalar multiplicatⁿ as for V.

Ex: Let $W_1 = \{(a, 0, 0) : a \in R\}$
$W_2 = \{(a, b, 0) : a, b \in R\}$
Here W_1 is subspace of W_2. Also W_1 & W_2 are subspace of R^3.

Linear combination :⇒

Let V be a vector space over a field F and let $v_1, v_2 --- v_n \in V$

Any vector of the form $a_1 \cdot v_1 + a_2 \cdot v_2 + --- + a_n \cdot v_n$ in V where $a_i \in F$ is called a linear combination of $v_1, v_2 --- v_n$

Linear dependence :⇒

Let V be a vector space over the field F. The vectors $v_1, v_2 --- v_n$ are said to be linearly dependent over F, if ∃ scalars $a_1, a_2 --- a_n \in F$ not all zero but linear combination is zero.

i.e $a_1 v_1 + a_2 v_2 + --- + a_n v_n = 0$
but all $a_i \neq 0$, where $i \in N$

Linear independence :⇒

Let V be a vector space over the field F. the vectors $v_1, v_2 --- v_n$ are said to be Linearly independent over F. if $\exists$ scalar $a_1, a_2 --- a_n \in F$ such that $a_1 v_1 + a_2 \cdot v_2 + --- + a_n \cdot v_n = 0 \Rightarrow$ all $a_i = 0$ where $i \in N$

*Linear span :⇒

Let S be a subset of the vector space V over the field F. The set of all linear combinations of vectors in S is called a linear span of S and is denoted by $L(S)$

If $S = \phi$, then $L(S) = 0$

Basis or Base of vector space V :⇒

Let V be a vector space over the field F. The set of vectors $\{v_1, v_2 --- v_n\}$ is called a basis of V, if

i) $v_1, v_2 --- v_n$ are linearly independent

ii) $v_1, v_2 -- v_n$ span V. i.e each vector of V can be uniquely expressed as Linear combination of $v_1, v_2, --- v_n$

Dimension of a vector space V :⇒

Number of Elements in a basis of vector space V is called the dimension of V and is denoted by $\dim V$.

If V contains a basis with n Elements then the $\dim V = n$

Note :⇒

* Two vectors v_1 & v_2 are linearly dependent iff one of them is a multiple of the other.

Ex: Determine whether or not the vectors v_1, v_2 are linearly dependent

i) $v_1 = (1, 3, 9)$. $v_2 = (2, 4, 1)$

→ No vector is the multiple of the other hence v_1, v_2 are not linearly dependent

ii) $v_1 = (3,4)$, $v_2 = (6,8)$

$\Rightarrow v_2 = 2v_1$ & hence v_1 & v_2 are linearly dependent vectors

problems :-

1) show that W is a subspace of $V(R)$ where $W = \{f : f(9) = 0\}$

Sol^n: since $0 \in W$ as $0(9) = 0$

so W is non empty set

Let $f, g \in W$.

i.e $f(9) = 0$ and $g(9) = 0$

then $\forall a, b \in R$

$(a \cdot f + b \cdot g) \cdot 9 = a \cdot f(9) + b \cdot g(9) = a \cdot 0 + b \cdot 0 = 0$

Hence $a \cdot f + b \cdot g \in W$.

By thm (2), W is a subspace of $V(R)$

Necessary and sufficient conditions for a subspace :-

Thm (1) :- W is a subspace of $V(F)$ iff

i) W is non empty

ii) W is closed under vector addition

i.e $\forall w_1, w_2 \in W \Rightarrow w_1 + w_2 \in W$

iii) W is closed under scalar multiplication

i.e $\forall a \in F$ and $w \in W \Rightarrow a \cdot w \in W$

Thm (2) :- W is a subspace of $V(F)$ iff

i) W is non empty

ii) $\forall a, b \in F$ and $v, w \in W$

$\Rightarrow a \cdot v + b \cdot w \in W$

problems :

1) Express the vector $v=(1,-2,5)$ as a linear combination of the vectors $v_1=(1,1,1)$, $v_2=(1,2,3)$ $v_3=(2,-1,1)$ in vector space $R^3(R)$

Sol: Let $v = a_1 v_1 + a_2 v_2 + a_3 v_3$; $a_1, a_2, a_3 \in R$

$(1,-2,5) = a_1(1,1,1) + a_2(1,2,3) + a_3(2,-1,1)$

$(1,-2,5) = (a_1+a_2+2a_3 ,\ a_1+2a_2-a_3 ,\ a_1+3a_2+a_3)$

Equating the corresponding Elements we get

$a_1+a_2+2a_3=1$, $a_1+2a_2-a_3=-2$, $a_1+3a_2+a_3=5$

solving above Eq^n we get

$a_1=-6$, $a_2=3$, $a_3=2$

Hence $(1,-2,5) = -6(1,1,1) + 3(1,2,3) + 2(2,-1,1)$

2) write the vector $v=(1,3,9)$ as a linear combination of the vectors $u_1=(2,1,3)$, $u_2=(1,-1,1)$ $u_3=(3,1,5)$

Sol: Let $v = xu_1 + yu_2 + zu_3$

$(1,3,9) = x(2,1,3) + y(1,-1,1) + z(3,1,5)$

$\Rightarrow 2x+y+3z=1$ $\quad x-y+z=3$ $\quad 3x+y+5z=9$

consider $AX=B$

$A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & -1 & 1 \\ 3 & 1 & 5 \end{bmatrix}$ $\quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ $\quad B = \begin{bmatrix} 1 \\ 3 \\ 9 \end{bmatrix}$

Augumented matrix

$[A:B] = \left[\begin{array}{ccc:c} 2 & 1 & 3 & 1 \\ 1 & -1 & 1 & 3 \\ 3 & 1 & 5 & 9 \end{array}\right]$

$R_1 \leftrightarrow R_2$

$[A:B] = \left[\begin{array}{ccc:c} 1 & -1 & 1 & 3 \\ 2 & 1 & 3 & 1 \\ 3 & 1 & 5 & 9 \end{array}\right]$

$R_2 \rightarrow R_2 - 2R_1$

$R_3 \rightarrow R_3 - 3R_1$

$$[A:B] = \begin{bmatrix} 1 & -1 & 1 & : & 3 \\ 0 & 3 & 1 & : & -5 \\ 0 & 4 & 2 & : & 0 \end{bmatrix}$$

$R_3 \rightarrow R_3 - R_2$

$$[A:B] = \begin{bmatrix} 1 & -1 & 1 & : & 3 \\ 0 & 3 & 1 & : & -5 \\ 0 & 1 & 1 & : & 5 \end{bmatrix}$$

$R_3 \leftrightarrow R_2$

$$[A:B] = \begin{bmatrix} 1 & -1 & 1 & : & 3 \\ 0 & 1 & 1 & : & 5 \\ 0 & 3 & 1 & : & -5 \end{bmatrix}$$

$R_3 \rightarrow R_3 - 3R_2$

$$[A:B] = \begin{bmatrix} 1 & -1 & 1 & : & 3 \\ 0 & 1 & 1 & : & 5 \\ 0 & 0 & -2 & : & -20 \end{bmatrix}$$

$\therefore$ Rank $[A]$ = Rank $[A:B] = 3$ = Number of unknowns

The system of linear eqn is consistent and possesses a unique soln

$x \cancel{+} y + z = 3$, $y + z = 5$, $-2z = -20$

$\therefore z = 10$, $y = -5$, $x = -12$

$(1, 3, 9) = -12(2, 1, 3) - 5(1, -1, 1) + 10(3, 1, 5)$

2) Determine whether the vectors $v_1 = (1, 2, 3)$ $v_2 = (3, 1, 7)$ and $v_3 = (2, 5, 8)$ are linearly dependent or linearly independent

soln. $x v_1 + y v_2 + z v_3 = 0$

$x(1, 2, 3) + y(3, 1, 7) + z(2, 5, 8) = (0, 0, 0)$

$x + 3y + 2z = 0$, $2x + y + 5z = 0$, $3x + 7y + 8z = 0$

Let $AX = 0$ where $A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 1 & 5 \\ 3 & 7 & 8 \end{bmatrix}$

$R_2 \rightarrow R_2 - 2R_1$

$R_3 \rightarrow R_3 - 3R_1$

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 0 & -5 & 1 \\ 0 & -2 & 2 \end{bmatrix}$$

$R_2 \rightarrow -R_2$

$R_3 \rightarrow \frac{-R_3}{2}$

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 5 & -1 \\ 0 & 1 & -1 \end{bmatrix}$$

$R_2 \leftrightarrow R_3$

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & -1 \\ 0 & 5 & -1 \end{bmatrix}$$

$R_3 \rightarrow R_3 - 5R_2$

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 4 \end{bmatrix}$$

Rank $[A] = 3 =$ Number of unknowns.

$\Rightarrow$ Homogeneous system of linear eq^n posses unique sol^n; $\Rightarrow$ $x = 0,\ y = 0,\ z = 0$

So vectors v_1, v_2, v_3 are linearly independent.

3) Determine whether the vectors $v_1 = (1, 4, 9)$, $v_2 = (3, 1, 4)$, $v_3 = (9, 3, 12)$ are linearly dependent or lineary independent.

Sol^n.- $x v_1 + y v_2 + z v_3 = 0$

$x(1, 4, 9) + y(3, 1, 4) + z(9, 3, 12) = (0, 0, 0)$

$x + 3y + 9z = 0, \quad 4x + y + 3z = 0 \quad 9x + 4y + 12z = 0$

Consider $AX = 0$ where

$$A = \begin{bmatrix} 1 & 3 & 9 \\ 4 & 1 & 3 \\ 9 & 4 & 12 \end{bmatrix}$$

$R_2 \rightarrow R_2 - 4R_1$

$R_3 \rightarrow R_3 - 9R_1$

$$= \begin{bmatrix} 1 & 3 & 9 \\ 0 & -11 & -33 \\ 0 & -23 & -69 \end{bmatrix}$$

$R_2 \rightarrow \frac{-R_2}{11}$ $R_3 \rightarrow \frac{-R_3}{23}$

$$= \begin{bmatrix} 1 & 3 & 9 \\ 0 & 1 & 3 \\ 0 & 1 & 3 \end{bmatrix}$$

$R_3 \rightarrow R_3 - R_2$

$$= \begin{bmatrix} 1 & 3 & 9 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

Rank $[A] = 2 < 3$ (Number of unknown)
so one unknown is constant & hence not always zero so that vectors are linearly dependent. or. $V_3 = 3V_2$

~~5Y~~ 07 or MQP ~~4~~ P.T in $V_3(R)$ the vectors $\{(1,2,1),(2,1,0),(1,-1,2)\}$ are linearly independent

Sol^n:- $V = a(1,2,1)x + (2,1,0)y + (1,-1,2)z$

$(0,0,0) =$

$x + 2y + z = 0$

$2x + y - z = 0$

$x + 2z = 0$

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & -1 \\ 1 & 0 & 2 \end{bmatrix} \Rightarrow \begin{matrix} R_2 \rightarrow R_2 - 2R_1 \\ R_3 \rightarrow R_3 - R_1 \end{matrix}$$

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -3 & -3 \\ 0 & -2 & 1 \end{bmatrix} \quad R_2 \rightarrow -\frac{1}{3}R_2$$

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & -2 & 1 \end{bmatrix}$$

$\cancel{R_2 \leftrightarrow R_3}$

$R_3 \rightarrow R_3 + 2R_2$

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 3 \end{bmatrix}$$

Rank of A = 3 = number of unknowns

Homogeneous system of linear Eq^n posses a unique sol^n $x=0,\ y=0,\ z=0$

i.e $x+2y+z=0$

$y+z=0$

$3z=0$

$z=0,\ y=0\quad x=0.$

$\therefore$ The given vectors are linearly independent

5> Express the vector $(3,5,2)$ as a linear combination of the vectors $(1,1,0),(2,3,0),(0,0,1)$ of $V_3(R)$

sol^n: Let $V = a_1v_1 + a_2v_2 + a_3v_3,\quad a_1,a_2,a_3 \in R$

$(3,5,2) = a_1(1,1,0) + a_2(2,3,0) + a_3(0,0,1)$

$(3,5,2) = (a_1+2a_2+\cancel{a_3},\ a_1+3a_2,\ a_3)$

Equating the corresponding elements

$a_1+2a_2 = \cancel{3}2$

$a_1+3a_2 = 5$

$a_3 = 2$

$\cancel{a_1}+3a_2 = 5$

$\cancel{a_1}+2a_2 = 3$

$(-)\quad(-)\quad(-)$

$a_2 = 2$

$a_1+3(2) = 5$

$a_1 = 5-6 = -1$

$a_1 = -1,\quad a_2 = 2,\quad a_3 = 2$

$(3,5,2) = -1(1,1,0) + 2(2,3,0) + 2(0,0,1)$

6> Are the vectors $V_1=(2,5,3)$, $V_2=(1,1,1)$ and $V_3=(4,-2,0)$ linearly independent? Justify your answer.

$V = a_1 v_1 + a_2 v_2 + a_3 v_3$

$(0,0,0) = a_1(2,5,3) + a_2(1,1,1) + a_3(4,-2,0)$

$(0,0,0) = (2a_1 + a_2 + 4a_3,\ 5a_1 + a_2 - 2a_3,\ 3a_1 + a_2)$

$2a_1 + a_2 + 4a_3 = 0$

$5a_1 + a_2 - 2a_3 = 0$

$3a_1 + a_2 = 0$

$$A = \begin{bmatrix} 2 & 1 & 4 \\ 5 & 1 & -2 \\ 3 & 1 & 0 \end{bmatrix} \quad R_1 \to R_1.$$

6) Determine whether or not each of the following form a basis $x_1 = (1,2,9)$, $x_2 = (2,-3,4)$, $x_3 = (1,3,7)$, $x_4 = (2,4,8)$ in R^3

sol^n: Basis in R^3 must contain exactly 3 Elements.

x_1, x_2, x_3, x_4 doesnot form a basis in R^3

7) Determine whether or not each of the following forms a basis, $x_1 = (2,2,1)$, $x_2 = (1,3,7)$, $x_3 = (1,2,2)$ in R^3

sol^n: Three vectors in R^3 form a basis iff they are linearly independent

$x(2,2,1) + y(1,3,7) + z(1,2,2) = (0,0,0)$

$2x + y + z = 0, \quad 2x + 3y + 2z = 0, \quad x + 7y + 2z = 0$

let $AX = 0$ where

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 3 & 2 \\ 1 & 7 & 2 \end{bmatrix} \quad R_1 \leftrightarrow R_3$$

$$A = \begin{bmatrix} 1 & 7 & 2 \\ 2 & 3 & 2 \\ 2 & 1 & 1 \end{bmatrix} \quad R_2 \to R_2 - 2R_1,\ R_3 \to R_3 - 2R_1$$

$$A = \begin{bmatrix} 1 & 7 & 2 \\ 0 & -11 & -2 \\ 0 & -13 & -3 \end{bmatrix} \quad R_2 \to -R_2,\ R_3 \to -R_3$$

$$A = \begin{bmatrix} 1 & 7 & 2 \\ 0 & 11 & 2 \\ 0 & 13 & 3 \end{bmatrix} \quad R_3 \to R_3 - R_2$$

$$A = \begin{bmatrix} 1 & 7 & 2 \\ 0 & 11 & 2 \\ 0 & 2 & 1 \end{bmatrix} \quad R_3 \to R_3/2$$

$$A = \begin{bmatrix} 1 & 7 & 2 \\ 0 & 11 & 2 \\ 0 & 1 & 1/2 \end{bmatrix} \quad R_3 \leftrightarrow R_2$$

$$A = \begin{bmatrix} 1 & 7 & 2 \\ 0 & 1 & 1/2 \\ 0 & 11 & 2 \end{bmatrix} \quad R_3 \to R_3 - 11R_2$$

$$A = \begin{bmatrix} 1 & 7 & 2 \\ 0 & 1 & 1/2 \\ 0 & 0 & -7/2 \end{bmatrix}$$

Rank of $[A] = 3 =$ No. of unknown so unique sol^n
and sol^n is $x=0,\ y=0,\ z=0$
Hence vectors x_1, x_2, x_3 are linearly independent
and hence form a basis

27.6.7 MQP ~~84~~ Determine whether or not the vector $(1,1,2)$ $(1,2,5)$ $(5,3,4)$ form a basis of R^3

Sol^n: Three vectors in R^3 form a basis iff they are linearly independent

$$x(1,1,2) + y(1,2,5), + z(5,3,4) = (0,0,0)$$

$$x + y + 5z = 0 \qquad x + 2y + 3z = 0 \qquad 2x + 5y + 4z = 0$$

where $AX = 0$ where

$$A = \begin{bmatrix} 1 & 1 & 5 \\ 1 & 2 & 3 \\ 2 & 5 & 4 \end{bmatrix}$$

$R_2 \to R_2 - R_1 \qquad R_3 \to R_3 - 2R_1$

$$A = \begin{bmatrix} 1 & 1 & 5 \\ 0 & 1 & -2 \\ 0 & 3 & -6 \end{bmatrix}$$

$R_3 \to R_3 - 3R_2$

$$A = \begin{bmatrix} 1 & 1 & 5 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

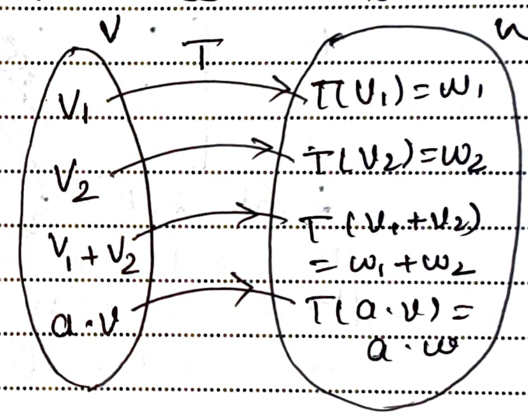
Rank of $A = 2 < 3$ Number of unknowns
So one unknown is constant and remaining two vectors are not multiples of each other
$\therefore$ It is Linearly independent.
The vectors form a basis of R^3

Linear Transformations :⇒

Let V and W be any two subspaces over a field F. A mapping T from V to W is called a linear transformation.

i) $V_1, V_2 \in V$
$T(V_1 + V_2) = T(V_1) + T(V_2)$

ii) $\forall\, a \in F$ and $\forall\, v \in V$
$T(a \cdot v) = a \cdot T(v)$



Probleme.

1) S.T the fun^n $T : R^2 \to R^3$ given by $T(x, y) = (x+y, x-y, y)$ is linear transformation

Sol^n: Let $u = (x_1, y_1)$, $v = (x_2, y_2)$ be two vectors belonging to R^2

$\therefore T(u+v) = T(x_1 + x_2, y_1 + y_2)$
$= \{x_1 + x_2 + y_1 + y_2,\ x_1 + x_2 - y_1 - y_2,\ y_1 + y_2)\}$
$= \{(x_1 + y_1) + (x_2 + y_2),\ (x_1 - y_1) + (x_2 - y_2),\ (y_1 + y_2)\}$
$= (x_1 + y_1,\ x_1 - y_1,\ y_1) + (x_2 + y_2,\ x_2 - y_2,\ y_2)$
$= T(u) + T(v)$

Also for $a \in R$ and $u \in R^2$ we have
$T(au) = T(ax_1, ay_1)$
$= (ax_1 + ay_1,\ ax_1 - ay_1,\ ay_1)$
$= a(x_1 + y_1,\ x_1 - y_1,\ y_1)$
$= a \cdot T(u)$

$\therefore$ T is a linear transformation.

2> Let $T: V(R) \longrightarrow V_2(R)$ be a mapping $f(x) = (3x, 5x)$ S.T f is a linear transformation

Sol^n: i> $f(x_1 + x_2) = [3(x_1+x_2), 5(x_1+x_2)]$
$= [3x_1 + 3x_2, 5x_1 + 5x_2]$
$= [3x_1, 5x_1] + [3x_2, 5x_2]$
$= f(x_1) + f(x_2)$

ii> $f(a \cdot x) = (3(a \cdot x), 5(a \cdot x))$
$= (3ax, 5ax)$
$= a(3x, 5x)$
$= a \cdot f(x)$

Hence f is a linear transformation.

Kernel of T and Image of T:

Let T be a linear transformation from V to W.
Kernel of T or $Ker(T) = \{ v \in V, T(v) = 0, 0 \in W \}$
Remark: Kernal of T is also known as null space of T.
Image of T or $Img(T) = \{ w \in W ; w = T(v)$ for some $v \in V \}$

Algebra of Linear transformations :-

+ Sum of linear transformations :-

Let $T_1 : U \rightarrow V$ and $T_2 : U \rightarrow V$ be two transformations where U & V are two vector spaces over the field F. we define the sum of T_1 & T_2 by $T_1 + T_2 : U \rightarrow V$ such that $(T_1 + T_2)u = T_1(u) + T_2(u)$, $u \in U$.

* Scalar multiplication of linear transformations :-

If $a \in F$, the fun^n aT i.e product of a linear transformation $T : U \rightarrow V$ with a, and defined by $(aT) : U \rightarrow V$, such that $(aT)u = a(T(u))$ for all $u \in U$ is linear transformation from U into V

* composition (product) of two L.T

Let u, v, w be three vector spaces over a field F. we define $T_2 . T_1$ called composition of or product of T_2 & T_1

by $(T_2T_1)(u) = T_2(T_1(u))$
$T_2 T_1$ is also denoted by $(T_2 o T_1)$

1) Let the linear transformations $T_1 : R^3 \to R^2$ such that $T_1(x,y,z) = (4x, 3y, 2z)$ and $T_2 : R^2 \to R^2$ such that $T_2(x,y) = (-2x, y)$ compute $T_1 T_2$ & $T_2 T_1$

Sol^n: As the range of T_2 is not contained in the domain of T_1

$\therefore$ $T_1 T_2$ is not defined.

Now, T_2T_1 is defined as the range of T_1 is contained in the domain of T_2

$T_2T_1(x,y,z) = T_2(T_1(x,y,z))$
$= T_2(4x, 3y-2z)$
$= (-8x, 3y, 2z)$

2) Illustrate with the help of an Example that there exist linear transformations $T_1 : R^2 \to R^2$ and $T_2 : R^3 \to R^2$ such that $T_2T_1 = 0$ but $T_1T_2 \neq 0$

Sol^n: Let us define $T_1 : R^2 \to R^2$ by
$T_1(x,y) = (0, 4x)$
& $T_2 : R^2 \to R^2$ by $T_2(x,y) = (x,0)$
$\therefore$ T_1T_2 are T_2T_1 both are defined.

$(T_2T_1)(x,y) = T_2(T_1(x,y)) = T_2(0, 4x)$
$= (0,0) = 0(x,y)$

$T_2T_1 = 0$

Also $(T_1T_2)(x,y) = T_1(T_2(x,y)) = T_1(x,0)$
$= (0, 4x) \neq 0(x,y)$

$T_1T_2 \neq 0$

3) Find the matrix representing the transformation $T : R^2 \to R^3$ given by $T(x_1, x_2) = (3x_1 - x_2, 2x_1 + 4x_2, 5x_1 - 6x_2)$ relative to the standard basis of R^2 and R^3

Sol^n: The ordered standard basis of R^2 is
$B = \{(1,0), (0,1)\}$ and that of R^3 is
$B' = \{(1,0,0), (0,1,0), (0,0,1)\}$

$\therefore T(x_1, x_2) = (3x_1 - x_2,\ 2x_1 + 4x_2,\ 5x_1 - 6x_2)$

$T(1,0) = (3,2,5)$ & $T(0,1) = (-1,4,-6)$

$T(1,0) = (3,2,5) = 3(1,0,0) + 2(0,1,0) + 5(0,0,1)$

$T(0,1) = (-1,4,-6) = -1(1,0,0) + 4(0,1,0) + (-6)(0,0,1)$

Hence, the matrix of the transformation is $\begin{bmatrix} 3 & -1 \\ 2 & 4 \\ 5 & -6 \end{bmatrix}$

4) Find the matrix representing the transformation $T: R^3 \to R^4$ defined by $T(x,y,z) = (x+y+z,\ 2x+z,\ 2y-z,\ 6y)$ relative to the standard basis of R^3 and R^4

Sol^n: w.r.t ordered standard basis of R^3 is $B = \{(1,0,0), (0,1,0), (0,0,1)\}$ and R^4 is $B' = \{(1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1)\}$

Since $T(x,y,z) = (x+y+z,\ 2x+z,\ 2y-z,\ 6y)$

$T(1,0,0) = (1,2,0,0)$

$T(0,1,0) = (1,0,2,6)$

$T(0,0,1) = (1,1,-1,0)$

Let $e_1 = (1,0,0,0)$, $e_2 = (0,1,0,0)$, $e_3 = (0,0,1,0)$, $e_4 = (0,0,0,1)$

$$\left.\begin{aligned} \therefore T(1,0,0) &= (1,2,0,0) = 1e_1 + 2e_2 + 0e_3 + 0e_4 \\ T(0,1,0) &= (1,0,2,6) = 1e_1 + 0e_2 + 2e_3 + 6e_4 \\ T(0,0,1) &= (1,1,-1,0) = 1e_1 + 1e_2 - 1e_3 + 0e_4 \end{aligned}\right\} \quad \text{—— (1)}$$

Matrix of T relative to B and B' is transpose of matrix of coefficients in above system of eq^n (1)

i.e $[T: B, B'] = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & 1 \\ 0 & 2 & -1 \\ 0 & 6 & 0 \end{bmatrix}$

5) Find the coordinates of vector $(1,1,1)$ relative to basis $(1,1,2), (2,2,1), (1,2,2)$

Sol^n: Let $(1,1,1) = a(1,1,2) + b(2,2,1) + c(1,2,2)$

$a + 2b + c = 1$ —— (1)

$a + 2b + 2c = 1$ —— (2)

$2a + b + 2c = 1$ —— (3)

Subtracting ① from ②, we have $c=0$
putting $c=0$ in ② and ③ we get

$a+2b=1$ —— ④

$2a+b=1$ —— ⑤

solving ④ & ⑤ we get $a=b=1/3$
∴ coordinates of $(1,1,1)$ relative to given basis are $(1/3, 1/3, 0)$

Rank and nullity of a linear operator :→

Rank of T :→
Dimension of the $Img(T)$ is known as rank of T.

Nullity of T :→
Dimension of the $Ker(T)$ is known as nullity of T.

Rank - nullity theorem :→
Let V and W be vector spaces over the field F. and let T be a linear transformation from V to W.
If V is a finite dimensional then
$rank(T) + nullity\ of\ (T) = dim(V)$

Problems :→

1) For the linear transformation $T: R^2 \to R^3$ such that $T(x_1, x_2) = (x_1 - x_2, x_2 - x_1, -x_1)$ find a basis and dimensi-ons of its range space and its null space Also verify that $rank(T) + nullity(T) = dim\ R^2$

sol^n: i) To find null space of T and its dimension.
by def^n, null space $= N(T) = \{u \in R^2, T(u) = 0 \in R^3\}$
Let $u = (x_1, x_2) \in R^2$ be an arbitrary element of null space

$T(u) = 0$

$T(x_1, x_2) = 0$

$(x_1 - x_2, x_2 - x_1, -x_1) = 0$

$\Rightarrow \quad x_1 - x_2 = 0$

$x_2 - x_1 = 0$

$-x_1 = 0$

on solving the above eq^n we get

$x_1 = 0, \; x_2 = 0$

Thus, the only number of null space is a zero vector $\in R^2$

i.e $N(T) = \{0\}$

$\therefore$ Nullity of $T = \dim(N(T)) = 0$

ii) To find range space of T and its dimension:

The range space consists of ordered triples $(x_1 - x_2, x_2 - x_1, -x_1)$ for $(x_1, x_2) \in R^2$ such that

$T(x_1, x_2) = (x_1 - x_2, x_2 - x_1, -x_1)$ ——— ①

Let v be any element of $R(T)$

Then there exist $(x_1, x_2) \in R^2$ such that $v = T(x_1, x_2)$

$\Rightarrow (x_1, x_2) = x_1(1,0) + x_2(0,1)$

$= x_1 e_1 + x_2 e_2$ where $e_1 = (1,0), e_2 = (0,1)$

$T(x_1, x_2) = T(x_1 e_1 + x_2 e_2)$

$= x_1 T(e_1) + x_2 T(e_2)$ ——— ②

$T(e_1) = T(1,0) = (1-0, 0-1, -1) = (1, -1, -1)$

$T(e_2) = T(0,1) = (0-1, 1-0, 0) = (-1, 1, 0)$

putting the values of $T(e_1), T(e_2)$ in ②

$T(x_1, x_2) = x_1(1,-1,-1) + x_2(-1,1,0)$

$v = x_1(1,-1,-1) + x_2(-1,1,0)$ ——— ③

Since $v \in R(T)$ is arbitrary.

iii) let $a(1,-1,-1) + b(-1,1,0) = 0$ for $a, b \in F$

$\Rightarrow (a-b, -a+b, -a) = (0,0,0)$

$\Rightarrow a-b = 0, \quad b-a = 0, \quad -a = 0$

$\Rightarrow \quad a = b = 0$

Thus $\{(1,-1,-1), (-1,1,0)\}$ is linearly independent, a

$\therefore$ It is a basis set of $R(T)$

$\therefore$ Dimension of $R(T) = 2$

Also dimension of $R^2 = 2$

$\therefore$ Rank(T) + nullity$(T) = 2 + 0 = 2$ (dim R^2)

27) Find a linear transformation $T: R^3 \rightarrow R^3$ whose range space is spanned by the vectors $(1,2,3), (4,5,6)$

Soln: w.k.t $\{e_1, e_2, e_3\}$ is a standard basis of R^3
where $e_1 = (1,0,0)$, $e_2 = (0,1,0)$. $e_3 = (0,0,1)$
Since R^3 is a three dimensional vector space and $\{e_1, e_2, e_3\}$ is a basis of R^3, there exist a unique linear transformation T such that

$T(e_1) = (1,2,3)$

$T(e_2) = (4,5,6)$

$T(e_3) = (0,0,0)$ (Assuming $T(0,0,1) = (0,0,0)$)

Also $R(T)$ is spanned by $\{(1,2,3), (4,5,6)\}$
i.e by $\{(1,2,3), (4,5,6), (0,0,0)\}$ or by $\{T(e_1), T(e_2), T(e_3)\}$
Now, for each $(x_1, x_2, x_3) \in R^3$

$(x_1, x_2, x_3) = x_1(1,0,0) + x_2(0,1,0) + x_3(0,0,1)$

$= x_1e_1 + x_2e_2 + x_3e_3$

28) Verify the Rank-nullity thm for the $T: R^3 \rightarrow R^3$ defined by $T(x,y,z) = (x+2y-z, y+z, x+y-2z)$

Soln: To find the basis of nullity of T.
Let $v = (x,y,z) \in R^3$ such that
Nullity of $T = \{v \in V / T(v) = 0\}$

$T(x,y,z) = 0$

$(x+2y-z, y+z, x+y-2z) = (0,0,0)$

$x+2y-z = 0 \Rightarrow x = 3z$

$y+z = 0 \Rightarrow y = -z$

$x+y-2z = 0$

On solving above eqn we get

$x = 3z, \quad y = -z$

$\{(x,y,z)\} = \{3z, -z, z\} = (z\{3,-1,1\})$

Thus $\{(3,-1,1)\}$ is a basis of nullity of T
& Nullity of $(T) = 1$

To find the basis of Range of T

As $\{(1,0,0), (0,1,0), (0,0,1)\}$ generates R^3

$\Rightarrow \{T(1,0,0), T(0,1,0), T(0,0,1)\}$ generates Range of T

$\Rightarrow \{(1,0,1), (2,1,1), (-1,1,-2)\}$ generates range of T

To find the basis of range

consider a matrix

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ -1 & 1 & -2 \end{bmatrix}$$

$R_2 \longrightarrow R_2 - 2R_1$

$R_3 \longrightarrow R_3 + R_2$

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{bmatrix}$$

$R_3 \longrightarrow R_3 - R_2$

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus, $\{(1,0,1), (0,1,-1)\}$ form a basis of Range of T

& dim (Range of T) = 2.

$\therefore$ Rank (T) + Nullity (T)

$= 2 + 1 = 3 = \dim(R^3)$

Hence Rank nullity thm verified.

Date .... Page No..........

Inner product spaces and orthogonality

Inner product space :⇒

let V be a real vector space. Suppose to each pair of vectors $u, v \in V$ there is assigned a real no. denoted by $\langle u, v \rangle$. This fun^n is called a inner product on V if it satisfies the following axioms

i) linear property :⇒ $\langle au_1 + bu_2, v \rangle = a\langle u_1, v \rangle + b\langle u_2, v \rangle$

ii) symmetric property :⇒ $\langle u, v \rangle = \langle v, u \rangle$

iii) Positive definite property : $\langle u, u \rangle \geq 0$ and

$\langle u, u \rangle = 0$ iff $u = 0$

Norm of a vector :⇒

An inner product $\langle u, u \rangle$ is non negative for any vector u

$\|u\| = \sqrt{\langle u, u \rangle}$ or $\|u\|^2 = \langle u, u \rangle$

This non negative number is called the norm or length of u

* $\hat{v} = \frac{1}{\|v\|} \cdot v$ (Normalizing v)

Orthogonality :⇒

let V be an inner product space. The vectors $u, v \in V$ are said to be orthogonal and u is said to be orthogonal to v if $\langle u, v \rangle = 0$

Date .... Page No...........

1) Determine the linear transformation $T: V_3(R) \to V_2(R)$ such that $T(e_1) = (-1,0)$, $T(e_2) = (1,1)$ and $T(e_3) = (0,-1)$ where e_1, e_2, e_3 is the standard basis of $V_3(R)$

Sol^n: let $(x,y,z) \in R^3$ be arbitrary

$(x,y,z) = a_1 v_1 + a_2 v_2 + a_3 v_3$

$a_1(1,0,0) + a_2(0,1,0) + a_3(0,0,1) = (x,y,z)$

$a_1 = x, \ a_2 = y, \ a_3 = z.$

we now have,

$(x,y,z) = x(1,0,0) + y(0,1,0) + z(0,0,1)$

Hence, $T(x,y,z) = x\,T(1,0,0) + y\,T(0,1,0) + z\,T(0,0,1)$

$T(x,y,z) = x\,T(e_1) + y\,T(e_2) + z\,T(e_3)$

$T(x,y,z) = x(-1,0) + y(1,1) + z(0,-1)$

Thus the required linear transformation

$T(x,y,z) = (y-x, y-z)$

2) If T_1, T_2 are linear operator from $V_2(R)$ to $V_2(R)$ defined by $T_1 = (x,y) = (x+y, x-y)$, $T_2(x,y) = (x-y, x+y)$ compute $T_1 \cdot T_2$, $T_2 T_1$, T_1^2, T_2^2

Sol^n: $\Rightarrow (T_1 T_2)(x,y) = T_1[T_2(x,y)] = T_1[x-y, x+y]$

$= [x-y+\overline{x+y}, \ x-y-\overline{x+y}]$

$T_1 T_2 = (2x, -2y)$ —— ①

Next $(T_2 \cdot T_1)(x,y) = T_2[T_1(x,y)] = T_2(x+y, x-y)$

$= [x+y-\overline{x-y}, \ x+y+\overline{x-y}]$

$T_2 \cdot T_1 = (2y, 2x)$ —— ②.

Comparing ① & ② $T_1 \cdot T_2 \neq T_2 \cdot T_1$ (Not commutative)

Also, $T_1^2(x,y) = T_1[T_1(x,y)] = T_1(x+y, x-y)$

$= (x+y+\overline{x-y}, \ x+y-\overline{x-y})$

$T_1^2 = (2x, 2y)$

Similarly $T_2^2 = (-2y, 2x)$

Teacher's Signature : ____________

1> Verify Rank - Nullity thm

$T: V_3(R) \longrightarrow V_2(R)$ defined by $T(x,y,z) = (y-x, y-z)$

$T(e_1) = T(1,0,0) = (-1,0)$

$T(e_2) = T(0,1,0) = (1,1)$

$T(e_3) = T(0,0,1) = (0,-1$

$[A] = \begin{bmatrix} -1 & 0 \\ 1 & 1 \\ 0 & -1 \end{bmatrix}$ $R_2 \to R_1 + R_2$, $R_3 \to R_2 + R_3$

Rank of $T = 2$

Nullity of $T = 1$

$2+3 = 3 = \dim(V_3(R))$ being 3

2> $T: V_3(R) \longrightarrow V_3(R)$ defined by $T(x,y,z) = (x+y, x-y, 2x+z)$

find range, null space, rank nullity in case of following LT

sol^n: $T(x,y,z) = (x+y, x-y, 2x+z)$

$T(1,0,0) = (1,1,2)$

$T(0,1,0) = (1,-1,0)$

$T(0,0,1) = (0,0,1)$

$[A] = \begin{bmatrix} 1 & 1 & 2 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $R_2 \to R_2 - R_1$, $R_2 \to -\frac{1}{2} R_1$

$[A] = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

Range $(T) = L(S)$

$= \{x_1(1,1,2) + x_2(0,1,1) + x_3(0,0,1)\}$

rank $= 3$ Nullity $= 0$

$3 + 0 = 3$ $\dim(V_3(R))$

$R(T) = \{(x_1, x_1 + x_2, 2x_1 + x_2 + x_3) / x_1, x_2, x_3 \in R\}$

Next consider $T(x,y,z) = 0$

$x+y = 0$, $x-y = 0$, $2x+z = 0$

on solving $x = 0, y = 0, z = 0$

$N(T) = \{(0,0,0)\}$

3> $T: V_3(R) \to V_3(R)$ defined by $T(x,y,z) = [x+2y+z, z-x, y+z]$