MQP 1) $e^{-3t} \sin 5t \cos 3t$ 1) $\frac{1-e^{t}}{t}$

2) $te^{-t}\sin 2t + \frac{\cos 2t - \cos 3t}{t}$

Soln $f(t) = \frac{\cos 2t - \cos 3t}{t}$

$$L[f(t)] = \int_s^\infty L(\cos 2t - \cos 3t)\,ds$$

$$= \int_s^\infty \left(\frac{s}{s^2+4} - \frac{s}{s^2+9}\right) ds$$

$$L[f(t)] = \left[\frac{1}{2}\log(s^2+4) - \frac{1}{2}\log(s^2+9)\right]_s^\infty$$

$$= \log\left(\sqrt{s^2+4/s^2+9}\right]_s^\infty$$

$$= \log\sqrt{1+(4/s^2)/1+(9/s^2)}\Big]_{s=\infty} - \log\sqrt{s^2+4/s^2+9}\Big]$$

$$= \log 1 - \log\left(\frac{\sqrt{s^2+4}}{\sqrt{s^2+9}}\right)$$

$$= \log\left(\sqrt{\frac{s^2+9}{s^2+4}}\right)$$

$L[te^{-t}\sin 2t]$

$$L[e^{-t}\sin 2t] = \left(\frac{2}{s^2+4}\right)_{s \to s+1}$$

$$= \frac{2}{(s+1)^2+4} = \frac{2}{s^2+2s+5}$$

$$L[te^{-t}\sin 2t] = (-1)^1 \frac{d}{ds}\left(\frac{2}{s^2+2s+5}\right)$$

$$= -\left[\frac{0 - 2(2s+2+0)}{(s^2+2s+5)^2}\right] = \frac{4(s+1)}{(s^2+2s+5)^2}$$

$$= \frac{4(s+1)}{(s^2+2s+5)^2} + \log\sqrt{\frac{s^2+9}{s^2+4}}$$

⇒ Laplace transform of periodic function :-

Defn :- A function $f(t)$ is said to be a periodic function of period $T > 0$ if $f(t+nT) = f(t)$ where $n = 1, 2, 3$

Ex :- $\sin t$, $\cos t$ are periodic function of period 2π because $\sin(t+2n\pi) = \sin t$

$\cos(t+2n\pi) = \cos t$.

Theorem :- If $f(t)$ is a periodic function of period T, then

$$L[f(t)] = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\, dt$$

proof :- We have defn

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt = \int_0^\infty e^{-su} f(u)\, du$$

$$= \int_{u=0}^{T} e^{-su} f(u)\, du + \int_{u=T}^{2T} e^{-su} f(u)\, du + \cdots + \int_{u=nT}^{(n+1)T} e^{-su} f(u)\, du + \cdots$$

$$L[f(t)] = \sum_{n=0}^{\infty} \int_{u=nT}^{(n+1)T} e^{-su} f(u)\, du \quad \text{——— (1)}$$

Now put, $u = t + nT$, $du = dt$

If $u = nT$ then $t + nT = nT \Rightarrow t = 0$

$u = (n+1)T$ then $t + nT = nT + T \Rightarrow t = T$

$f(u) = f(t+nT) = f(t)$ by the periodic property

using these result in the RHS of (1)

$$L[f(t)] = \sum_{n=0}^{\infty} \int_{t=0}^{T} e^{-s(t+nT)} f(t)\, dt$$

$$= \sum_{n=0}^{\infty} e^{-snT} \int_{t=0}^{T} e^{-st} f(t)\, dt$$

But $\sum_{n=0}^{\infty} e^{-snT} = \sum_{n=0}^{\infty} (e^{-sT})^n = 1 + e^{-sT} + (e^{-sT})^2 + \cdots$

putting $r = e^{-sT}$

$$\sum_{n=0}^{\infty} e^{-snT} = \frac{1}{1-e^{-sT}}$$

Now using ③ in RHS of ② we have

$$L[f(t)] = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\,dt$$

1) If $f(t) = t^2$, $0<t<2$ and $f(t+2) = f(t)$ for $t>2$, find $L[f(t)]$

Sol^n: $f(t)$ is a periodic function of period $T=2$

we have $L[f(t)] = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\,dt$

$$L[f(t)] = \frac{1}{1-e^{-2s}} \int_0^2 e^{-st}\, t^2\,dt$$

$$= \frac{1}{1-e^{-2s}} \int_0^2 t^2 e^{-st}\,dt$$

Applying Bernoullis rule of integration by parts,

$$L[f(t)] = \frac{1}{1-e^{-2s}} \left\{ \left[t^2 \cdot \frac{e^{-st}}{-s}\right]_0^2 - \left[2t\, \frac{e^{-st}}{s^2}\right]_0^2 + \left[2 \cdot \frac{e^{-st}}{-s^3}\right]_0^2 \right\}$$

$$= \frac{1}{1-e^{-2s}} \left\{ \frac{-1}{s}(4e^{-2s}-0) - \frac{2}{s^2}(2e^{-2s}-0) - \frac{2}{s^3}(e^{-2s}-1) \right\}$$

$$L[f(t)] = \frac{2}{s^3(1-e^{-2s})} \left\{ -2s^2 e^{-2s} - 2s e^{-2s} - e^{-2s} + 1 \right\}$$

$$L[f(t)] = \frac{2}{s^3(1-e^{-2s})}\left\{1-(2s^2+2s+1)e^{-2s}\right\}$$

2) Find the Laplace transform of the periodic function defined by $f(t) = \frac{kt}{T}$, $0<t<T$; $f(t+T) = f(t)$

Soln:- We have $L[f(t)] = \frac{1}{1-e^{-sT}}\int_0^T e^{-st} f(t)\,dt$

$$L[f(t)] = \frac{k}{T(1-e^{-sT})}\int_0^T t e^{-st}\,dt$$

Applying Bernoulli's rule we have,

$$L[f(t)] = \frac{k}{T(1-e^{-sT})}\left[t\left(\frac{e^{-st}}{-s}\right) - 1\left(\frac{e^{-st}}{s^2}\right)\right]_0^T$$

$$= \frac{k}{T(1-e^{-sT})}\left[\left(\frac{-1}{s}Te^{-sT} - 0\right) - \frac{1}{s^2}\left(e^{-sT}-1\right)\right]$$

$$L[f(t)] = \frac{-ke^{-sT}}{s(1-e^{-sT})} + \frac{k}{s^2T}$$

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3) Find the Laplace transform of the full wave rectifier $f(t) = E\sin\omega t$, $0<t<\pi/\omega$ having period π/ω

Soln:-

$$L[f(t)] = \frac{1}{1-e^{-sT}}\int_0^T e^{-st} f(t)\,dt$$

NOW, $L[f(t)] = \frac{1}{1-e^{-s(\pi/\omega)}}\int_0^{\pi/\omega} e^{-st} E\sin\omega t\,dt$

$$= \frac{E}{1-e^{-\pi s/\omega}}\left[\frac{e^{-st}}{s^2+\omega^2}(-s\sin\omega t - \omega\cos\omega t)\right]_0^{\pi/\omega}$$

$$= \frac{E}{(1-e^{-\pi s/\omega})(s^2+\omega^2)}\left(e^{-\pi s/\omega}\cdot\omega - (-\omega)\right)$$

$$L[f(t)] = \frac{E\omega(1+e^{-\pi s/\omega})}{(s^2+\omega^2)(1-e^{-\pi s/\omega})}$$

We can simplify further

Multiplying both the Nr & Dr. by $e^{\pi s/2\omega}$ in RHS,

$$L[f(t)] = \frac{E\omega}{s^2+\omega^2} \frac{(e^{\pi s/2\omega} + e^{-\pi s/2\omega})}{(e^{\pi s/2\omega} - e^{-\pi s/2\omega})}$$

$$= \frac{E\omega}{s^2+\omega^2} \cdot \frac{2\cosh(\pi s/2\omega)}{2\sinh(\pi s/2\omega)}$$

$$L[f(t)] = \frac{E\omega}{s^2+\omega^2} \coth(\pi s/2\omega)$$

3) Given $f(t) = \begin{cases} E, & 0<t<a/2 \\ -E, & a/2<t<a \end{cases}$ where $f(t+a) = f(t)$

show that $L[f(t)] = E/s \tanh(as/4)$.

Soln:- The given function is periodic with period $T=a$

We have

$$L[f(t)] = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\, dt$$

$$= \frac{1}{1-e^{-sa}} \int_0^a e^{-st} f(t)\, dt$$

$$L[f(t)] = \frac{1}{1-e^{-as}} \left\{ \int_0^{a/2} e^{-st} E\, dt + \int_{a/2}^{a} e^{-st} (-E)\, dt \right\}$$

$$= \frac{E}{1-e^{-as}} \left\{ \left[\frac{e^{-st}}{-s}\right]_0^{a/2} + \left[\frac{e^{-st}}{s}\right]_{a/2}^{a} \right\}$$

$$= \frac{E}{s(1-e^{-as})} \cdot \left\{ -\left[e^{-st}\right]_0^{a/2} + \left[e^{-st}\right]_{a/2}^{a} \right\}$$

$$= \frac{E}{s(1-e^{-as})}\left\{-e^{-as/2}+1+e^{-as}-e^{-as/2}\right\}$$

$$= \frac{E}{s(1-e^{-as})}\left(1-2e^{-as/2}+e^{-as}\right)$$

$$= \frac{E(1-e^{-as/2})^2}{s(1-e^{-as})}$$

$$L[f(t)] = \frac{E(1-e^{-as/2})^2}{s(1-e^{-as/2})(1+e^{-as/2})} = \frac{E(1-e^{-as/2})}{s(1+e^{-as/2})}$$

Multiplying both the Nr. & Dr. by $e^{as/4}$ we get,

$$L[f(t)] = \frac{E(e^{as/4}-e^{-as/4})}{s(e^{as/4}+e^{-as/4})} = \frac{E.2\sinh(as/4)}{s.2\cosh(as/4)}$$

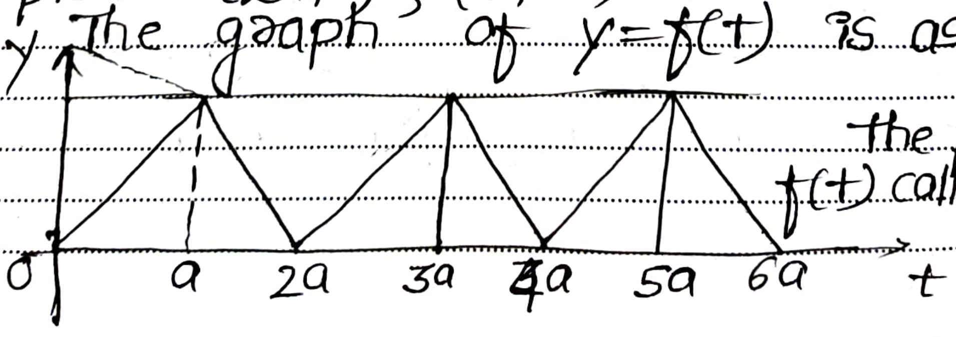
$$L[f(t)] = \frac{E}{s}\tanh\left(\frac{as}{4}\right)$$

MQ8 4) If $f(t) = \begin{cases} t, & 0\le t\le a \\ 2a-t, & a\le t\le 2a \end{cases}$, $f(t+2a) = f(t)$

Soln: i) sketch the graph of $f(t)$ as a periodic function
ii) show that $L[f(t)] = \frac{1}{s^2}\tanh(as/2)$

Soln- i) Let $f(t) = y$ and $y = t$ is st. line passing through the origin making an angle 45° with the t-axis. $y = 2a-t$ or $y+t = 2a$ or $t/2a + y/2a = 1$ is a st. line passing through the pts. $(2a,0)$ & $(0,2a)$

The graph of $y = f(t)$ is as follows.



the periodic funct. $f(t)$ called triangular wave function.

ii) We have $T = 2a$ and $L[f(t)] = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\, dt$

$$L[f(t)] = \frac{1}{1-e^{-2as}} \int_0^{2a} e^{-st} f(t)\, dt$$

$$= \frac{1}{1-e^{-2as}} \left\{ \int_0^a t\, e^{-st} dt + \int_a^{2a} (2a-t) e^{-st} dt \right\}$$

Applying Bernoulli's rule to each of the Integrals

$$L[f(t)] = \frac{1}{1-e^{-2as}} \left\{ \left[t \frac{e^{-st}}{-s} - (1) \frac{e^{-st}}{s^2} \right]_0^a + \left[(2a-t) \frac{e^{-st}}{-s} - (-1) \frac{e^{-st}}{s^2} \right]_a^{2a} \right\}$$

$$= \frac{1}{1-e^{-2as}} \left\{ -\frac{1}{s}(a e^{-as} - 0) - \frac{1}{s^2}(e^{-as} - 1) - \frac{1}{s}(0 - a e^{-as}) + \frac{1}{s^2}(e^{-2as} - e^{-as}) \right\}$$

$$L[f(t)] = \frac{1}{s^2(1-e^{-2as})} (-e^{-as} + 1 + e^{-2as} - e^{-as})$$

$$= \frac{1}{s^2(1-e^{-2as})} (1 - 2e^{-as} + e^{-2as})$$

$$= \frac{(1-e^{-as})^2}{s^2(1-e^{-as})(1+e^{-as})}$$

$$L[f(t)] = \frac{(1-e^{-as})}{s^2(1+e^{-as})} = \frac{e^{as/2} - e^{-as/2}}{s^2(e^{as/2} + e^{-as/2})}$$

where we have multiplied both the Nr. & Dr. by $e^{as/2}$

$$L[f(t)] = \frac{2\sinh(as/2)}{s^2 \cdot 2\cosh(as/2)} = \frac{1}{s^2}\tanh\left(\frac{as}{2}\right)$$

$$L[f(t)] = \frac{1}{s^2}\tanh\left(\frac{as}{2}\right)$$

* Similar problems.

June 2015 i) Find $L[f(t)]$ if $f(t) = \begin{cases} t, & 0 \le t \le 1 \\ 2-t, & 1 \le t \le 2 \end{cases}$

June 2018 ii) Find $L[f(t)]$ if $f(t) = \begin{cases} t, & 0 < t < \pi \\ \pi - t & \pi < t < 2\pi \end{cases}$

5) A periodic function of period $2\pi/\omega$ is defn by $f(t) = \begin{cases} E\sin\omega t & 0 \le t \le \pi/\omega \\ 0, & \pi/\omega \le t \le 2\pi/\omega \end{cases}$ where E & ω are constant.

Show that $L[f(t)] = \dfrac{E\omega}{(s^2+\omega^2)(1-e^{-\pi s/\omega})}$

Soln:- We have for a periodic function $f(t)$

$$L[f(t)] = \frac{1}{1-e^{-sT}}\int_0^T e^{-st} f(t)\,dt$$

Here, $T = 2\pi/\omega$

$$L[f(t)] = \frac{1}{1-e^{-2\pi s/\omega}}\int_0^{2\pi/\omega} e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{-2\pi s/\omega}}\left\{\int_0^{\pi/\omega} e^{-st} E\sin\omega t\,dt + \int_{\pi/\omega}^{2\pi/\omega} e^{-st}\cdot 0\,dt\right.$$

$$= \frac{E}{1-e^{-2\pi s/\omega}}\left[\frac{e^{-st}}{(-s)^2+\omega^2}(-s\sin\omega t - \omega\cos\omega t)\right]_0^{\pi/\omega}$$

$$= \frac{-E}{(s^2+\omega^2)(1-e^{-2\pi s/\omega})}\left\{e^{-s\pi/\omega}(s\sin\pi + \omega\cos\pi) - e^0(s\sin 0 + \omega\cos 0)\right\}$$

$$=$$

$$= \frac{-E}{(s^2+\omega^2)(1-e^{-2\pi s/\omega})}\left\{-\omega e^{-s\pi/\omega}-\omega\right\}$$

$$= \frac{E\omega(1+e^{-\pi s/\omega})}{(s^2+\omega^2)(1-e^{-2\pi s/\omega})}$$

$$L[f(t)] = \frac{E\omega(1+e^{-\pi s/\omega})}{(s^2+\omega^2)(1-e^{-\pi s/\omega})(1+e^{-\pi s/\omega})}$$

$$L[f(t)] = \frac{E\omega}{(s^2+\omega^2)(1-e^{-\pi s/\omega})}$$

* Unit step function:-

Defn:- The unit step function $u(t-a)$ or Heaviside function $H(t-a)$ is defined as follows.

$$u(t-a) = \begin{cases} 0, & t \leq a \\ 1, & t > a \end{cases}$$ where a is constant.

* properties associated with the unit step function.

i) $L[u(t-a)] = \frac{e^{-as}}{s}$

ii) $L[f(t-a)\,u(t-a)] = e^{-as}\bar{f}(s)$ where $L[f(t)] = \bar{f}(s)$

iii) If $f(t) = \begin{cases} f_1(t), & t \leq a \\ f_2(t), & t > a \end{cases}$

then $f(t) = f_1(t) + [f_2(t) - f_1(t)]\,u(t-a)$

⇒ Find the laplace transform of the following functions.

1) $f(t) = \begin{cases} t, & 0 < t < 4 \\ 5, & t > 4 \end{cases}$

$f(t) = t + (5-t)\,u(t-4)$

Soln:- $f(t) = t + (5-t)\,u(t-4)$

$L[f(t)] = L(t) + L[(5-t)\,u(t-4)]$

We have, $L(t) = 1/s^2$

$F(t-4) = (5-t)$

$F(t) = 5-(t+4) = 1-t$

$\bar{F}(s) = L[f(t)] = \frac{1}{s} - \frac{1}{s^2}$

$$L[F(t-4)u(t-4)] = e^{-4s}\bar{F}(s)$$

$$L[(5-t)u(t-4)] = e^{-4s}\left(\frac{1}{s}-\frac{1}{s^2}\right)$$

we shall use the results in

$$L[f(t)] = \frac{1}{s^2} + e^{-4s}\left(\frac{1}{s}-\frac{1}{s^2}\right)$$

$$= \frac{1}{s}e^{-4s} + \frac{1}{s^2}\left(1-e^{-4s}\right)$$

2) $f(t) = \begin{cases} \sin 2t, & 0<t<\pi \\ 0 & t>\pi \end{cases}$

sol:- $f(t) = \sin 2t + (0-\sin 2t)u(t-\pi)$ by a property

$$L[f(t)] = L(\sin 2t) - L[(\sin 2t)u(t-\pi)]$$

$L[\sin 2t\, u(t-\pi)]$

taking $F(t-\pi) = \sin 2t$,

$F(t) = \sin 2(t+\pi) = \sin(2\pi+2t)$

$F(t) = \sin 2t$

$$\bar{F}(s) = 2/s^2+4$$

$$L[F(t-\pi)u(t-\pi)] = e^{-\pi s}\bar{F}(s) = 2e^{-\pi s}/s^2+4$$

We shall use the result in ①

$$L[f(t)] = \frac{2}{s^2+4} - \frac{2e^{-\pi s}}{s^2+4} = \frac{2(1-e^{-\pi s})}{s^2+4}$$

MQP

* using unit step function, find the LT of

$f(t) = \begin{cases} \cos t, & 0\le t\le\pi \\ \cos 2t, & \pi<t\le 2\pi \\ \cos 3t & t\ge 2\pi \end{cases}$

3) $f(t) = \begin{cases} \cos t, & 0 < t < \pi \\ \sin t & t > \pi \end{cases}$

Sol:- $f(t) = \cos t + (\sin t - \cos t)\, u(t-\pi)$ by a property

$L[f(t)]$ $F(t) = \sin(t+\pi) - \cos(t+\pi)$

$= -\sin t + \cos t$

$\bar{F}(s) = \frac{-1}{s^2+1} + \frac{s}{s^2+1} + \frac{s-1}{s^2+1}$

But $L[F(t-\pi)\, u(t-\pi)] = e^{-\pi s} \bar{F}(s) = \frac{e^{-\pi s}(s-1)}{s^2+1}$

We shall use these results in (1)

Thus, $L[f(t)] = \frac{s}{s^2+1} + \frac{e^{-\pi s}(s-1)}{s^2+1}$

$= \frac{s + e^{-\pi s}(s-1)}{s^2+1}$

4) $f(t) = \begin{cases} 1, & 0 < t \leq 1 \\ t & 1 < t \leq 2 \\ t^2 & t > 2 \end{cases}$

Sol:- $f(t) = 1 + (t-1)\, u(t-1) + (t^2 - t)\, u(t-2)$ by a property

$L[f(t)] = L(1) + L[(t-1)\, u(t-1)] + L(t^2-t)\, u(t-2)]$

Let $F(t-1) = (t-1)$; $G(t-2) = t^2 - t$

$F(t) = t$, $G(t) = (t+2)^2 - (t+2) = t^2 + 3t + 2$

$\bar{F}(s) = \frac{1}{s^2}$, $\bar{G}(s) = \frac{2}{s^3} + \frac{3}{s^2} + \frac{2}{s}$

$L[F(t-1)\, u(t-1)] = e^{-s} \bar{F}(s)$ &

$L[G(t-2)\, u(t-2)] = e^{-2s} \bar{G}(s)$

$L[(t-1)\, u(t-1)] = \frac{e^{-s}}{s^2}$ &

$L[(t^2-t)\, u(t-2)] = e^{-2s}\left(\frac{2}{s^3} + \frac{3}{s^2} + \frac{2}{s}\right)$

$L[f(t)] = \frac{1}{s} + \frac{e^{-s}}{s^2} + e^{-2s}\left(\frac{2}{s^3} + \frac{3}{s^2} + \frac{2}{s}\right)$

* Inverse Laplace Transforms:-

Inverse Laplace Transform can be regarded as the reverse process of finding the laplace Transform of a given function.

→ We also discuss convolution theorem which helps in finding the inverse Laplace Transform.

If $L[f(t)] = \bar{f}(s)$

where $f(t)$ is called inverse laplace transform $\bar{f}(s)$ & denoted as.

$L^{-1}[\bar{f}(s)] =$

thus $L[f(t)] = \bar{f}(s)$

$L^{-1}[\bar{f}(s)] = f(t)$

$L(1) = \frac{1}{s} \Rightarrow L^{-1}\left(\frac{1}{s}\right) = 1$

$L(\cos at) = \frac{s}{s^2+a^2} \Rightarrow L^{-1}\left(\frac{s}{s^2+a^2}\right) = \cos at$

⇒ Table of LT of std. function given earlier to present the basic table of Inverse Laplace transform.

	Function	Inverse Transform		Function	Inverse transform
1)	$\frac{1}{s}$	1	5	$\frac{1}{s^2+a^2}$	$\frac{\sin at}{a}$
2.	$\frac{1}{s-a}$	e^{at}		$\frac{1}{s^2-a^2}$	$\frac{\sinh at}{a}$
3)	$\frac{s}{s^2+a^2}$	$\cos at$		$\frac{1}{s^{n+1}}$	$\frac{t^n}{\Gamma(n+1)}$
4)	$\frac{s}{s^2-a^2}$	$\cosh at$		$\frac{1}{s^{n+1}}$, $n=1,2,3$	$\frac{t^n}{n!}$

1) $L^{-1}\left(\frac{1}{s-1}\right) = e^{t}$

2) $L^{-1}\left(\frac{s}{s^2+9}\right) = \cos 3t$

3) $L^{-1}\left(\frac{1}{s^2+5}\right) = \frac{1}{\sqrt{5}}\sin(\sqrt{5}t)$

4) $L^{-1}\left(\frac{1}{s^4}\right) = \frac{t^3}{3!}$

5) $L^{-1}\left(\frac{1}{s+1}\right) = e^{-t}$

6) $L^{-1}\left(\frac{s}{s^2-16}\right) = \cosh 4t$

7) $L^{-1}\left(\frac{1}{s^2-36}\right) = \frac{1}{6}\sinh 6t$

8) $L^{-1}\left(\frac{1}{s^{3/2}}\right) = \frac{t^{1/2}}{\Gamma(3/2)} = \frac{t^{1/2}}{\frac{1}{2}\sqrt{\pi}} = 2\sqrt{\frac{t}{\pi}}$

* Find the inverse Laplace Transform of the following functions.

1) $\frac{1}{s+2} + \frac{3}{2s+5} - \frac{4}{3s-2}$

Sol^n:- $L^{-1}\left[\frac{1}{s+2}\right] + \frac{3}{2}\left[L^{-1}\left[\frac{1}{s+5/2}\right]\right] - \frac{4}{3}L^{-1}\left[\frac{1}{s-2/3}\right]$

Thus the required inverse Laplace transform is

$$e^{-2t} + \frac{3}{2}e^{-5t/2} - \frac{4}{3}e^{2t/3}$$

2) $L^{-1}\left[\frac{s}{s^2+6^2}\right] + 2L^{-1}\left[\frac{1}{s^2+6^2}\right] + 4L^{-1}\left[\frac{s}{s^2+5^2}\right] - L^{-1}\left[\frac{1}{s^2+5^2}\right]$

Sol^n:- Thus the required inverse Laplace transform is

$\cos 6t + \frac{1}{3}\sin 6t + 4\cos 5t - \frac{1}{5}\sin 5t$

3) $2L^{-1}\left[\frac{s}{4s^2+25}\right] - 5L^{-1}\left[\frac{1}{4s^2+25}\right] + 8L^{-1}\left[\frac{1}{16s^2+9}\right] - 6L^{-1}\left[\frac{s}{16s^2+9}\right]$

$= \frac{1}{2}L^{-1}\left[\frac{s}{s^2+(5/2)^2}\right] - \frac{5}{4}L^{-1}\left[\frac{1}{s^2+(5/2)^2}\right] + \frac{1}{2}L^{-1}\left[\frac{1}{s^2+(3/4)^2}\right]$

$- \frac{3}{8}L^{-1}\left[\frac{s}{s^2+(3/4)^2}\right]$

$= \frac{1}{2}\cos\left(\frac{5t}{2}\right) - \frac{5}{4}\cdot\frac{2}{5}\sin\left(\frac{5t}{2}\right) + \frac{1}{2}\times\frac{4}{3}\sin\left(\frac{3t}{4}\right) - \frac{3}{8}\cos\left(\frac{3t}{4}\right)$

$= \frac{1}{2}\cos\left(\frac{5t}{2}\right) - \frac{1}{2}\sin\left(\frac{5t}{2}\right) + \frac{2}{3}\sin\left(\frac{3t}{4}\right) - \frac{3}{8}\cos\left(\frac{3t}{4}\right)$

4) $L^{-1}\left[\frac{2s-5}{2(4s^2-25)}\right]-L^{-1}\left[\frac{4s}{s^2-9}\right]$

$=\frac{1}{2}L^{-1}\left[\frac{1}{2s+5}\right]-4L^{-1}\left[\frac{s}{s^2-3^2}\right]$

$=\frac{1}{2}\times\frac{1}{2}L^{-1}\left[\frac{1}{s+(5/2)}\right]-4L^{-1}\left[\frac{s}{s^2-3^2}\right]$

$=\frac{1}{4}e^{-5t/2}-4\cosh 3t$

5) $L^{-1}\left[\frac{s^3+6s^2+12s+8}{s^6}\right]$

soln:- $=L^{-1}\left(\frac{1}{s^3}\right)+6L^{-1}\left(\frac{1}{s^4}\right)+12L^{-1}\left(\frac{1}{s^5}\right)+8L^{-1}\left(\frac{1}{s^6}\right)$

$=\frac{t^2}{2!}+6\cdot\frac{t^3}{3!}+12\frac{t^4}{4!}+8\frac{t^5}{5!}$

$=\frac{t^2}{2}+t^3+\frac{t^4}{2}+\frac{t^5}{15}$

6) $\frac{3}{2}L^{-1}\left[\frac{s^4-2s^2+1}{s^5}\right]$

soln:- $=\frac{3}{2}\left[L^{-1}\left(\frac{1}{s}\right)-2L^{-1}\left(\frac{1}{s^3}\right)+L^{-1}\left(\frac{1}{s^5}\right)\right]$

$=\frac{3}{2}\left[1-2\frac{t^2}{2!}+\frac{t^4}{4!}\right]=\frac{3}{2}\left[1-t^2+\frac{t^4}{24}\right]$

7) $3L^{-1}\left[\frac{s}{s^2+(\sqrt{8})^2}\right]+5\sqrt{2}L^{-1}\left[\frac{1}{s^2+(\sqrt{8})^2}\right]$

soln:- $3\cos(2\sqrt{2}t)+5/2\sin(2\sqrt{2}t)$

8) $L^{-1}\left[\frac{1}{s^{3/2}}\right] + 3L^{-1}\left[\frac{1}{s^{5/2}}\right] - 8L^{-1}\left[\frac{1}{s^{1/2}}\right]$

Soln:-
$$= \frac{t^{1/2}}{\Gamma(3/2)} + \frac{3t^{3/2}}{\Gamma(5/2)} - \frac{8t^{-1/2}}{\Gamma(1/2)}$$

$$= \frac{\sqrt{t}}{\frac{1}{2}\Gamma(\frac{1}{2})} + \frac{3t\sqrt{t}}{\frac{3}{2}\times\frac{1}{2}\Gamma(\frac{1}{2})} - \frac{8}{\sqrt{t}\,\Gamma(\frac{1}{2})}$$

$$= \frac{2\sqrt{t}}{\sqrt{\pi}} + \frac{4t\sqrt{t}}{\sqrt{\pi}} - \frac{8}{\sqrt{t}\sqrt{\pi}}$$

$$= \frac{2}{\sqrt{\pi}}\left[\sqrt{t} + 2t\sqrt{t} - \frac{4}{\sqrt{t}}\right]$$

* Computation of the inverse transform of $e^{-as}\bar{f}(s)$

We have proved that
$$L[f(t)] = \bar{f}(s)$$
$$L[f(t-a)u(t-a)] = e^{-as}\bar{f}(s)$$
$$L^{-1}[e^{-as}\bar{f}(s)] = f(t-a)\,u(t-a)$$

Working procedure

Step 1) Given funct. we should observe the presence of e^{-as} first & identify the remaining part of the function to be called as $\bar{f}(s)$

Step 2) Taking the inverse of $\bar{f}(s)$ we obtain $f(t)$

Step 3) The required inverse of $e^{-as}\bar{f}(s)$ is obtained by replacing t by $(t-a)$ in $f(t)$ to be multiplied by the unit step function $u(t-a)$

1) $\frac{1+e^{-3s}}{s^2}$

Sol:- $L^{-1}\left(\frac{1}{s^2}\right) + L^{-1}\left(\frac{e^{-3s}}{s^2}\right)$

$L^{-1}\left(\frac{1}{s^2}\right) = t$

$L^{-1}\left(\frac{e^{-3s}}{s^2}\right) = (t-3)\,u(t-3)$

$= t + (t-3)\,u(t-3)$

2) $3L^{-1}\left[\frac{1}{s^2}\right] + 2L^{-1}\left(\frac{e^{-s}}{s^3}\right) - 3L^{-1}\left(\frac{e^{-2s}}{s}\right)$

Sol: We have $L^{-1}\left(\frac{1}{s^2}\right) = t$, $L^{-1}\left(\frac{1}{s^3}\right) = \frac{t^2}{2}$, $L^{-1}\left(\frac{1}{s}\right) = 1$

$3t + 2\frac{(t-1)^2}{2}u(t-1) - 3\times 1\times u(t-2)$

thus, $L^{-1}\left[\frac{3}{s^2} + \frac{2e^{-s}}{s^3} - \frac{3e^{-2s}}{s}\right]$

$= 3t + (t-1)^2 u(t-1) - 3u(t-2)$

3) $\frac{\cosh 2s}{e^{3s}s^2} = \frac{e^{-3s}}{s^2} \cdot \frac{(e^{2s}+e^{-2s})}{2} = \frac{1}{2}\left[\frac{e^{-s}}{s^2} + \frac{e^{-5s}}{s^2}\right]$

Now, $L^{-1}\left[\frac{\cosh 2s}{e^{3s}s^2}\right] = \frac{1}{2}\left\{L^{-1}\left(\frac{e^{-s}}{s^2}\right) + L^{-1}\left(\frac{e^{-5s}}{s^2}\right)\right\}$

But $L^{-1}\left(1/s^2\right) = t$

Thus: $L^{-1}\left[\frac{\cosh 2s}{e^{3s}s^2}\right] = \frac{1}{2}\left\{(t-1)\,u(t-1) + (t-5)\,u(t-5)\right\}$

4) $\frac{(1-e^{-s})(2-e^{-2s})}{s^3} = \frac{2-2e^{-s}-e^{-2s}+e^{-3s}}{s^3}$

Sol:- Now, $L^{-1}\left[\frac{(1-e^{-s})(2-e^{-2s})}{s^3}\right]$

$= 2L^{-1}\left(\frac{1}{s^3}\right) - 2L^{-1}\left(\frac{e^{-s}}{s^3}\right) - L^{-1}\left(\frac{e^{-2s}}{s^3}\right) + L^{-1}\left(\frac{e^{-3s}}{s^3}\right)$

But $L^{-1}\left(\frac{1}{s^3}\right) = t^2/2$

Thus, $L^{-1}\left[\frac{(1-e^{-s})(2-e^{-2s})}{s^3}\right]$ is given by

$= t^2 - (t-1)^2 u(t-1) - \frac{(t-2)^2 u(t-2)}{2} + \frac{(t-3)^2 u(t-3)}{2}$

$= t^2 - (t-1)^2 u(t-1) - \frac{(t-2)^2 u(t-2)}{2} + \frac{(t-3)^2 u(t-3)}{2}$

* $L^{-1}\left[\frac{e^{-\pi s}}{s^2+1}\right] + L^{-1}\left[e^{-2\pi s}\frac{s}{s^2+4}\right]$

soln:- We have $L^{-1}\left(\frac{1}{s^2+1}\right) = \sin t$,

$L^{-1}\left[\frac{s}{s^2+4}\right] = \cos 2t$

Hence ① becomes,

$\sin(t-\pi)\, u(t-\pi) + \cos 2(t-2\pi)\, u(t-2\pi)$

thus, $L^{-1}\left[\frac{e^{-\pi s}}{s^2+1} + \frac{se^{-2\pi s}}{s^2+1}\right] = -\sin t\, u(t-\pi) + \cos 2t\, u(t-2\pi)$

* $L^{-1}\left(e^{-s/2}\frac{s}{s^2+\pi^2}\right) + L^{-1}\left(e^{-s}\frac{\pi}{s^2+\pi^2}\right)$

$L^{-1}\left[\frac{s}{s^2+\pi^2}\right] = \cos \pi t$, $L^{-1}\left(\frac{\pi}{s^2+\pi^2}\right) = \sin \pi t$

Hence ① becomes

$\cos \pi(t-1/2)\, u(t-1/2) + \sin \pi(t-1)\, u(t-1)$

$= \sin \pi t\, u(t-1/2) - \sin \pi t\, u(t-1)$

Thus, $L^{-1}\left[\frac{se^{-s/2} + \pi e^{-s}}{s^2+\pi^2}\right] = \sin \pi t\left[u(t-1/2) - u(t-1)\right]$

* Inverse transform by completing the squares:-

We have the property that
$L[f(t)] = \bar{f}(s)$ then $L[e^{at} f(t)] = \bar{f}(s-a)$

this implies that
$$L^{-1}[\bar{f}(s)] = f(t)$$
$$L^{-1}[\bar{f}(s-a)] = e^{at} f(t)$$
$$L^{-1}[\bar{f}(s-a)] = e^{at} L^{-1}[\bar{f}(s)]$$

Working procedure for problems

Step 1:- Given $\bar{f}(s) = \dfrac{\phi(s)}{(ps^2+qs+r)}$

We first express (ps^2+qs+r) in the form $(s-a)^2 \pm b^2$ & later express $\phi(s)$ in terms of $(s-a)$ so that the given function of square reduces to a function of $(s-a)$

Step 2:- We then use 3 to obtain the result

Step 3:- However if $\bar{f}(s) = \phi(s)/\psi(s-a)$ we only need to express $\phi(s)$ in terms of $(s-a)$ to compute the inverse transform.

Find inverse LT of the following functions

1) $\frac{s+5}{s^2-6s+13}$

Sol^n:- $L^{-1}\left[\frac{s+5}{(s-3)^2+4}\right] = L^{-1}\left[\frac{s-3+3+5}{(s-3)^2+2^2}\right]$

$= L^{-1}\left[\frac{(s-3)+8}{(s-3)^2+2^2}\right]$

Here $a=3$ and $s-3$ changes to s

$= e^{3t}\left\{L^{-1}\left[\frac{s+8}{s^2+2^2}\right]\right.$

$= e^{3t}\, L^{-1}\left[\frac{s}{s^2+2^2}\right] + 8\, L^{-1}\left(\frac{1}{s^2+2^2}\right)$

$= e^{3t}(\cos 2t + 4\sin 2t)$

2) $\frac{s^2}{(s+1)^3}$

Sol^n:- We need to express s^2 in terms of $(s+1)$

$s^2 = (s+1)^2 - 2s - 1 = (s+1)^2 - 2(s+1) + 2 - 1$

$s^2 = (s+1)^2 - 2(s+1) + 1$

$L^{-1}\left[\frac{s^2}{(s+1)^3}\right] = L^{-1}\left[\frac{(s+1)^2 - 2(s+1) + 1}{(s+1)^3}\right]$

Here, $a=-1$, and $(s+1)$ changes to s

hence RHS becomes

$e^{-t} L^{-1}\left[\frac{s^2-2s+1}{s^3}\right] = e^{-t}\left\{L^{-1}\left[\frac{1}{s}\right] - 2L^{-1}\left(\frac{1}{s^2}\right) + L^{-1}\left(\frac{1}{s^3}\right)\right.$

$L^{-1}\left[\frac{s^2}{(s+1)^3}\right] = e^{-t}\left(1 - 2t + \frac{t^2}{2}\right)$

Here, $a=-1$ and $(s+1)$ changes to s

Hence RHS becomes,

$e^{-t} L^{-1}\left[\frac{s^2-2s+1}{s^3}\right] = e^{-t}\left\{L^{-1}\left(\frac{1}{s}\right) - 2L^{-1}\left(\frac{1}{s^2}\right) + L^{-1}\left(\frac{1}{s^3}\right)\right.$

$$L^{-1}\left[\frac{s^2}{(s+1)^3}\right] = e^{-t}\left(1-2t+\frac{t^2}{2}\right)$$

3) $\frac{(s+2)e^{-s}}{(s+1)^4}$

Sol:- We shall first find $L^{-1}[\bar{f}(s)] = f(t)$

$$L^{-1}\left[\frac{s+2}{(s+1)^4}\right] = L^{-1}\left[\frac{(s+1)+1}{(s+1)^4}\right] = e^{-t} L^{-1}\left[\frac{s+1}{s^4}\right]$$

$$L^{-1}\left[\frac{(s+2)}{(s+1)^4}\right] = e^{-t}\left\{L^{-1}\left[\frac{1}{s^3}\right] + L^{-1}\left[\frac{1}{s^4}\right]\right\}$$

Using $L^{-1}\left[\frac{1}{s^{n+1}}\right] = \frac{t^n}{n!}$, we get

$$f(t) = e^{-t}\left(\frac{t^2}{2!}+\frac{t^3}{3!}\right) = e^{-t}\left(\frac{t^2}{2}+\frac{t^3}{6}\right)$$

$$L^{-1}[e^{-s}\bar{f}(s)] = f(t-1)u(t-1)$$

$$L^{-1}\left[e^{-s}\frac{s+2}{(s+1)^4}\right] = e^{-(t-1)}\left\{\frac{(t-1)^2}{2}+\frac{(t-1)^3}{6}\right\}u(t-1)$$

4) $$L^{-1}\left[\frac{2s-1}{s^2+4s+29}\right] = L^{-1}\left[\frac{2s-1}{(s+2)^2+25}\right]$$

$$= L^{-1}\left[\frac{2(s+2)-5}{(s+2)^2+25}\right]$$

$$= e^{-2t} L^{-1}\left[\frac{(2s-5)}{s^2+5^2}\right] = e^{-2t}\left\{2L^{-1}\left[\frac{s}{s^2+5^2}\right]\right.$$

$$= 2L^{-1}\left[\frac{s}{s^2+5^2}\right] - L^{-1}\left[\frac{5}{s^2+5^2}\right]$$

$$= L^{-1}\left[\frac{2s-1}{s^2+4s+29}\right] = e^{-2t}(2\cos 5t - \sin 5t)$$

5) Find inverse LT. i) $\frac{(s^2-1)^2}{s^5}$ ii) $\frac{s}{s^2+6s+13}$

$$L^{-1}\left[\frac{s^4+1-2s^2}{s^5}\right] = L^{-1}\left[\frac{1}{s} + \frac{1}{s^5} - \frac{2}{s^3}\right]$$

$$= 1 + \frac{t^4}{24} - \frac{2t^2}{2}$$

$$= 1 + \frac{t^4}{24} - t^2$$

$$L^{-1}\left[\frac{(s+3)-3}{(s+3)^2+4}\right] = e^{-3t} L^{-1}\left[\frac{s-3}{s^2+4}\right]$$

$$= e^{-3t}\left[\cos 2t - \frac{3}{2}\sin 2t\right]$$

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5) $L^{-1}\left[\frac{s+1}{s^2+6s+9}\right] = L^{-1}\left[\frac{(s+3)-2}{(s+3)^2}\right]$

$= e^{-3t} L^{-1}\left[\frac{s-2}{s^2}\right]$

$= e^{-3t}\left[L^{-1}\left(\frac{1}{s}\right) - 2L^{-1}\left(\frac{1}{s^2}\right)\right\}$

$= L^{-1}\left[\frac{s+1}{s^2+6s+9}\right] = e^{-3t}(1-2t)$

6) $\frac{e^{-4s}}{(s-4)^2}$

$L^{-1}[\bar{f}(s)] = L^{-1}\left[\frac{1}{(s-4)^2}\right] = e^{4t} L^{-1}\left[\frac{1}{s^2}\right] = e^{4t}\, t = f(t)$

$L^{-1}[e^{-4s}\bar{f}(s)] = f(t-4)\,u(t-4)$

$L^{-1}\left[\frac{e^{-4s}}{(s-4)^2}\right] = e^{4(t-4)}(t-4)\,u(t-4)$

7) $\frac{s+1}{s^2+s+1}$

sol:- $s^2+s+1 = \left(s+\frac{1}{2}\right)^2 - \frac{1}{4} + 1 = \left(s+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2$

$L^{-1}\left[\frac{s+1}{s^2+s+1}\right] = L^{-1}\left[\frac{(s+1/2)+1/2}{(s+1/2)^2+(\sqrt{3}/2)^2}\right]$

$= e^{-t/2} L^{-1}\left[\frac{s}{s^2+(\sqrt{3}/2)^2}\right] + \frac{1}{2} L^{-1}\left[\frac{1}{s^2+(\sqrt{3}/2)^2}\right]$

$= e^{-t/2} L^{-1}\left[\frac{s+1}{s^2+s+1}\right] = e^{-t/2}\left\{\cos\left(\sqrt{3}t/2\right) + \frac{1}{\sqrt{3}}\sin\left(\sqrt{3}t/2\right)\right\}$

8) $e^{-2s}\left(\frac{s+4}{s^2+8s+20}\right)$

$s^2+8s+16+4$

$e^{-s}\frac{s}{(s^2-6s+18}$

$(s-3)^2+$

* Inverse transform by the method of partial fractions:-

WKT, the method of partial fraction is a technique of converting an algebraic fraction $\phi(s)/\psi(s)$
→ [where degree of $\phi(s)$ is less than that of $\psi(s)$] into a sum. Depending on the nature of terms in $\psi(s)$ we have to split into a sum of various terms with constant A, B, C, D - - which can be determined later the inverse is found term by term.

Find inverse LT

Ex:-1) $\frac{1}{s(s+1)(s+2)(s+3)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2} + \frac{D}{s+3}$

Multiplying by $s(s+1)(s+2)(s+3)$ we get

$$1 = A(s+1)(s+2)(s+3) + B(s+2)s(s+3) + Cs(s+1)(s+3) + Ds(s+1)(s+2)$$

put $s=0$: $1 = A(6)$ $\therefore A = 1/6$

put $s=-1$: $1 = B(-2)$ $B = -1/2$

put $s=-2$: $1 = C(2)$ $C = 1/2$

put $s=-3$: $1 = D(-6)$ $D = -1/6$

Now, $L^{-1}\left[\frac{1}{s(s+1)(s+2)(s+3)}\right]$

$$= \frac{1}{6}L^{-1}\left[\frac{1}{s}\right] - \frac{1}{2}L^{-1}\left[\frac{1}{s+1}\right] + \frac{1}{2}L^{-1}\left[\frac{1}{s+2}\right] - \frac{1}{6}L^{-1}\left[\frac{1}{s+3}\right]$$

$$= L^{-1}\left[\frac{1}{s(s+1)(s+2)(s+3)}\right] = \frac{1}{6} - \frac{1}{2}e^{-t} + \frac{1}{2}e^{-2t} - \frac{1}{6}e^{-3t}$$

2) $\dfrac{2s^2+5s-4}{s(s-1)(s+2)}$

Sol^n $\dfrac{2s^2+5s-4}{s(s-1)(s+2)} = \dfrac{A}{s} + \dfrac{B}{(s-1)} + \dfrac{C}{s+2}$

Multiplying $s(s-1)(s+2)$ we get

$2s^2+5s-4 = A(s-1)(s+2) + Bs(s+2) + Cs(s-1)$

put $s=0 \quad -4 = A(-2) \quad \therefore A = 2$

$s=1 \quad 3 = B(3) \quad B = 1$

$s=-2 \quad -6 = C(6) \quad C = -1$

Now, $L^{-1}\left[\dfrac{2s^2+5s-4}{s(s-1)(s+2)}\right]$

$= 2L^{-1}\left[\dfrac{1}{s}\right] + L^{-1}\left[\dfrac{1}{s-1}\right] - L^{-1}\left[\dfrac{1}{s+2}\right]$

$= 2 + e^{t} - e^{-2t}$

3) $\dfrac{3s+2}{s^2-s-2}$

Sol^n- Let $\dfrac{3s+2}{(s-2)(s+1)} = \dfrac{A}{(s-2)} + \dfrac{B}{(s+1)}$

$3s+2 = A(s+1) + B(s-2)$

put $s=2$, $8 = A(3) \quad A = 8/3$

$s=-1$, $-1 = B(-3) \quad B = 1/3$

$L^{-1}\left[\dfrac{3s+2}{(s-2)(s+1)}\right] = \dfrac{8}{3}L^{-1}\left[\dfrac{1}{s-2}\right] + \dfrac{1}{3}L^{-1}\left[\dfrac{1}{s+1}\right]$

$= L^{-1}\left[\dfrac{3s+2}{s^2-s-2}\right] = \dfrac{1}{3}(8e^{2t}+e^{-t})$

4) $\frac{5s+3}{(s-1)(s^2+2s+5)} = \frac{A}{(s-1)} + \frac{Bs+c}{s^2+2s+5}$

Sol $5s+3 = A(s^2+2s+5) + (Bs+c)(s-1)$

put $s=1$, $8 = A(8) \Rightarrow A=1$

put $s=0$, $3 = 5A-c$ $c=2$

Equating the coefficient of s^2 on both sides of ① we get

$0 = A+B$

$B = -1$

Now, we have

$\frac{5s+3}{(s-1)(s^2+2s+5)} = \frac{1}{s-1} + \frac{-s+2}{(s^2+2s+5)} = \frac{1}{(s-1)} + \frac{2-s}{(s+1)^2+4}$

$\frac{5s+3}{(s-1)(s^2+2s+5)} = \frac{1}{(s-1)} + \frac{3-(s+1)}{(s+1)^2+4}$

$L^{-1}\left[\frac{5s+3}{(s-1)(s^2+2s+5)}\right] = L^{-1}\left[\frac{1}{s-1}\right] + L^{-1}\left[\frac{3-(s+1)}{(s+1)^2+4}\right]$

$= e^t + e^{-t} L^{-1}\left[\frac{3-s}{s^2+4}\right]$

$= e^t + e^{-t}\left\{3L^{-1}\left[\frac{1}{s^2+4}\right] - L^{-1}\left[\frac{s}{s^2+4}\right]\right\}$

thus, $L^{-1}\left[\frac{(5s+3)}{(s-1)(s^2+2s+5)}\right] = e^t + e^{-t}\left[\frac{3}{2}\sin 2t - \cos 2t\right]$

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5) Find ILT $\frac{2s^2-6s+5}{s^3-6s^2+11s-6} = \frac{2s^2-6s+5}{(s-1)(s-2)(s-3)}$

$A = \frac{1}{2}$, $B = -1$, $c = \frac{5}{2}$

$= \frac{e^t}{2} - e^{2t} + \frac{5}{2}e^{3t}$